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KAPLANSKY TEST PROBLEM FOR R-MODULES

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ABSTRACT
We prove that every ring R without strong decomposition theorem has a strong non-decomposition theorem. We use diamonds (but this will be eliminated in a subsequent paper).

§1. Introduction

R will be a ring, not necessarily commutative, with 1; R-module is a left R-module unless stated otherwise. In [Sh54] = [Sh54a] 8.7 we proved

1.A. THEOREM. *For every ring R, either:*

(1) *all R-modules are the direct sum of countably generated R-modules (such rings are called left pure semisimple rings)*

or

(2) *for every cardinal $\lambda > |R|$,*

$(2)_\lambda$ *there is an R-module M of power λ such that for no $\mu < \lambda$ is M the direct sum of R-modules of power $\leq \mu$.*

In fact $(1) \Leftrightarrow \neg(2) \Leftrightarrow$ *the class of R-modules is superstable* $\Leftrightarrow$ *a condition on equations in R.*

Subsequently, Garavaglia [Gr] and then Ziegler [Z] much improve the results concerning (1) (e.g., unique decomposition to indecomposable modules). See more in Prest [P1] and [P2] about the history of this and other equivalent conditions.

But here we want to strengthen possibility (2); more specifically, we want to

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show for case (2) there are R-modules which have few endomorphisms, are "rigid like", and, moreover, that the decomposition theory for R-modules is "bad"; e.g., that the answer to:

$$M \cong N \oplus M_1, \qquad N \cong M \oplus N_1 \Rightarrow N \cong M?$$

(Kaplansky's first test problem) is negative.

In a classical way we do it by giving a ring S (the ring of endomorphisms we want) and try to build an R-module which "has the endomorphisms for $s \in S$ but not many more".

The literature on the endomorphism of modules (including the restriction to indecomposability or rigidity, and to abelian groups which are exactly the $\mathbb{Z}$-modules) is quite large.

Kaplansky in [K] suggested test problems for having a satisfactory decomposition theory.

Fuchs, with some help of Corner, proved the existence of an indecomposable abelian group in many cardinals λ (e.g., up to the first strongly inaccessible) [Fu], and even of a system of 2^λ rigid abelian groups of power λ (the proof was by induction on λ). In fact it seems at the time reasonable that for some "large cardinal" (e.g., supercompact) this fails. Corner [C2] reduced the number of primes to five.

Shelah [Sh44] proved the existence in every λ (using stationary sets). Lately, Gobel and Ziegler generalized this to R-modules for "R with five ideals". Shelah [Sh45], answering a question of Pierce, constructed reduced separable (abelian) p-groups with only "small" + p-adic endomorphism but has to use λ strong limit of uncountable cofinality.

Eklof and Mekler [EM], using diamond on λ† (and a non-reflecting stationary set) got a λ-free indecomposable abelian group of power λ; continuing this, in [Sh140] the diamond was replaced by weak diamond on a non-reflecting stationary subset of $S = \{\delta < \lambda : \mathrm{cf}\,\delta = \aleph_0\}$ (so for $\lambda = \aleph_1$, $2^{\aleph_0} < 2^{\aleph_1}$ suffices).

Much earlier Corner [C] proved that we can realize any torsion-free reduced countable ring as an endomorphism ring of a torsion-free abelian group and deduce by it a negative answer to, e.g., the Kaplansky problem cited above.

Dugas [D1] continuing [EM] proved the existence of a strongly κ-free abelian group with endomorphism ring Z (if, e.g., $V = L$) and then Gobel [G1] realized a larger family of rings; he used p-adic rings.

Dugas and Gobel [DG1], continuing [D1], [G1] and [Sh140] (but [DG1] used one

†It is a consequence of $V = L$ but not provable in ZFC.

prime), for λ as in [Sh140], proved: for a ring R of cardinality $< \lambda$, which is cotorsion free, i.e. $(R,+)$ (an additive group) is torsion free, reduced and contain no direct summands isomorphic to I_p (p-adic completion of $\mathbf{Z}$) for all primes p. Dugas and Gobel [DG2] characterize the rings which can be represented as End M modulo "the small endomorphism" for some abelian p-group, but as it continues [Sh45] (which dealt with the case when we want the smallest such ring) the representation of a ring R is by an abelian group M of a power strong limit cardinal of cofinality $> |R|$. The situation is similar in Dugas and Gobel [DG3] where the results of [GD1] and more are obtained in such cardinals.

In [Sh172] + [Sh227] we introduce a principle "black box", which follows from ZFC, that enables us to get the results of [DG2], [DG3] in more and smaller cardinals, e.g., $\lambda = (|R|^{\aleph_0})^+$.

Corner and Gobel [CG] continue this; see there and in [EM1] for additional references.

In 2.1–2.5 we give the algebraic setting and choose specific bimodules which we will use.

Next, 2.6 is the diamond construction (with a non-reflecting stationary set $\mathcal{S} \subseteq \{\delta < \lambda : \mathrm{cf}\,\delta = \aleph_0\}$, with $\diamondsuit_{\mathcal{S}}$). The construction is phrased such that its existence is immediate.

Main fact 2.7 tells us that every R-endomorphism of M_λ (the bimodule constructed in 2.6) is somewhat definable.

However, we later use an even slightly weaker variant defined in 2.8(3), $(\mathrm{Pr}^-)^{n(*)}_\alpha[F]$ (some $\alpha < \lambda$, $n(*) < \omega$). In 2.10 we show that it implies a stronger version $((\mathrm{Pr}\ 1)^{n(*)}_{\alpha,z})$. The rest of the section explicates the result: in M_λ every endomorphism is in some sense equal to one in a ring dE. The ring dE depends on R and S (but not on λ); the "in some sense equal" means: for each n we restrict F to a sub-abelian group $\varphi_n(M_\lambda)$ (closed under F), divide by another $(\bigcap_l \varphi_l(M_\lambda))$ and take the direct limit; on top of this we have an "error term": we have to divide by a "small" submodule of M_λ, which means of cardinality $< \lambda$. An alternative presentation is: we divide the ring of such endomorphisms by the ideal of those with "small" range.

In section 3 we try to make the "error term" smaller. We have to avoid a "large member" of $\mathcal{K}$ (e.g., projectives). So we fix a family of bimodules $\mathcal{K}$ (e.g., those which are finitely generated, finitely presented). Then we ask M_λ to be λ-free in a sense; i.e., where $M_\lambda = \bigcup_{\alpha<\lambda} M_\alpha$, M_α increasing continuous of power $< \lambda$, demand that every M_α is the direct sum of members of $\mathcal{K}$. We get this time a smaller error term—its power is $\leq |R| + |S| + \aleph_0$ and, if R, S are countable, it disappears.

In section 4 we draw specific consequences of our representation theorem.

In a subsequent paper [Sh421] we get the main results in ZFC (without any extra axioms); this is as done originally. We lose the λ-freeness (this is unavoidable, even for abelian groups — see Magidor and Shelah [MgSh204]). We also get, for each $m(*)$, an R-module M such that $M \cong M^n$ iff n divides $m(*)$ and the other Kaplanksy test problems. We shall also point out that the theorems apply to elementary (= first order) classes of modules which are not totally transcendental.

We thank Gobel and Ziegler for helpful questions on an earlier version of the work.

1.B. Remark. We use $\langle N_n, N'_n, N^{\rm tr}_n, g_n : n < \omega \rangle$ (see 2.5) totally determined by $\langle \varphi_n : n < \omega \rangle$ (and T, R, S). However, we do not use all their specific properties, just:

(a) N_n a bimodule with a distinguished element $x^{[n]}$.

(b) g_n is a (bimodule) homomorphism from N_n to N_{n+1} mapping $x^{[n]}$ to $x^{[n+1]}$.

(c) Let $\varphi_n(M)$ be defined as

$$\{h(x^{[n]}) : h \text{ an } R\text{-homomorphism from } N_n \text{ to } M\}.$$

(d) There is no R-homomorphism h from N_{n+1} to $N_n, x^{[n+1]}h = x^{[n]}$.

(e) f^1_n, f^2_n are R-homomorphisms from N_n to $N'_n, x^{[n]}f^1_n = x^{[n]}f^2_n$, $N'_n = \operatorname{Rang} f'_n$ and

$$N^{\rm tr}_n = \left\{ yf^1_n : y \in N_n,\ yf^1_n - yf^2_n \text{ belongs to } \bigcap_{m<\omega} \varphi_m(N'_n) \right\}.$$

§2. The diamond construction

2.1. Remark. If you want to deal with many $\bar{\varphi}$'s simultaneously, no change is required.

2.2. Context and Fact. (a) R, S rings with unit 1, T a commutative subring of Cent R and of Cent S (Cent — the center). A bimodule M is a left R-module, right S-module such that $(rx)s = r(xs)$, $tx = xt$ for $x \in M$, $t \in T$, $r \in R$, $s \in S$ (really we should say an (R, S)-bimodule). T, R and S are fixed here (except in §4). K, M, N denote bimodules (or left R-modules).

Homomorphisms (f, g, h, F), particularly of R-modules, should be written from the right (so composition is accordingly). They are homomorphism of bimodules if not said otherwise; an R-homomorphism has the obvious meaning.

(b) The class of (R, S)-bimodules is a variety. For a homomorphism $M_1 \xrightarrow{F} M_2$, the kernel $\operatorname{Ker} F = \{x \in M_0 : xF = 0\}$ is a sub-bimodule of M_1, and the image, $\operatorname{Rang} F = \{xF : x \in M_1\}$, is a sub-bimodule of M_2; F preserves the satisfaction of p.e. (= positive existential) formulas.

(c) If $M_1 \subseteq M_2$ (M_1 a sub-bimodule of M_2) then $M_2/M_1 = \{x + m_1 : x \in M_1\}$ is a homomorphic image of $M_2, x \mapsto x + M_1$ a homomorphism, with kernel M_1.

2.3. Assumption. For some bimodule M^* and sequence $\bar{\varphi} = \langle \varphi_n(x) : n < \omega \rangle$ of conjunctive positive existential formulas (in the language of left R-modules, see below):

$\langle \varphi_n(M^*) : n < \omega \rangle$ is strictly decreasing where $\varphi_n(M) = \{x \in M : M \vDash \varphi_n[x]\}$.

[By [Sh54] 8.7 it exists if possibility (1) of Theorem 1.A fails.]

2.3A. Observation. $\varphi_n(M^*)$ is a subgroup of M^* as an (additive) group and even a sub-right S-module, but not necessarily a sub-bimodule.

2.4. Trivial Derivations from the Assumption. Let

$$\varphi_n(x) = (\exists y_0, \cdots, y_{q_n - 1})\left(\bigwedge_{l=0}^{m_n - 1} a_l^n x = \sum_{i < k_l^n} b_{l,i}^n y_i\right),$$

so $a_l^n, b_{l,i}^n$ are members of R.

As we can replace φ_n by $\bigwedge_{l \le n} \varphi_l$, interchange order of $\exists$ and $\bigwedge$ and change names of variables without loss of generality: $k_l^n = k_l$, $a_l^n = a_l$, $b_{l,i}^n = b_{l,i}$, $k_l < k_{l+1}$, $m_n < m_{n+1}$, and also without loss of generality $m_0 = 1$, $a_0 = 1_R$, $k_0 = 1$, $b_{0,0} = 1$; i.e., $\varphi_0(x) = \exists y_0 (x = y_0)$ and $q_n = k_{m_n - 1}$.

2.5. Definition and Claim. (a) Let N_n be the bimodule generated freely by $\{x\} \cup \{y_i : 0 \le i < k_{m_n - 1}\}$ subject only to the equations $\{a_l x = \sum_{i < k_l} b_{l,i} y_i : l < m_n\}$. When confusion may arise we write $x^{[n]}, y_i^{[n]}$.

(b) Trivially: $x \in \varphi_n(N_n)$.

(c) Trivially: if M is a bimodule, *then* $x^* \in \varphi_n(M)$ iff for some homomorphism h from N_n into M as bimodules, $xh = x^*$.

(d) By the choice of M^* and $\bar{\varphi}$ (and 2.5(c) above): $x \notin \varphi_{n+1}(N_n)$.

(e) Let N_n' be freely generated by x, y_i', y_i'' for $i < k_{m_n - 1}$ subject only to the relations:

$$a_l x = \sum_{i < k_l} b_{l,i} y_i',$$

$$a_l x = \sum_{i < k_n} b_{l,i} y_i''.$$

Let N_n^ζ for $\zeta = 1,2$ be the sub-bimodule of N_n' generated by:

$$\{x\} \cup \{y_i' : i < k_{m_n - 1}\} \qquad \text{for } \zeta = 1,$$

$$\{x\} \cup \{y_i'' : i < k_{m_n - 1}\} \qquad \text{for } \zeta = 2.$$

Let $f_n^\zeta : N_n \xrightarrow{f_n^\zeta} N_n^\zeta$ be the bimodule homomorphism defined by: $x f_n^\zeta = x$; $y_i f_n^1 = y_i'$, $y_i f_n^2 = y_i''$.

(f) $N_n^{\mathrm{tr}} = \{z \in \varphi_n(N_n) : z f_n^1 - z f_n^2 \in \bigcap_l \varphi_l(N_n')\}$ is an abelian subgroup of N_n (and S-submodule, as $\bigcap_l \varphi_l(N_n'')$ is).

2.6. The Construction. Here we give the simpler variant, under diamond, sufficient for Kaplansky test problems.

We let $|R| + |S| + \aleph_0 < \lambda = \operatorname{cf} \lambda$, $\mathcal{S} \subseteq \{\delta < \lambda : \operatorname{cf} \delta = \aleph_0\}$ is stationary but does not reflect, $\diamondsuit_{\mathcal{S}}$, without loss of generality $\mathcal{S}^* = \{\delta < \lambda : \operatorname{cf} \delta = \aleph_0,\ \delta \notin \mathcal{S}\}$ is stationary too. We define, by induction on $\alpha \leq \lambda$, M_α such that:

(A) M_α is a bimodule and has universe $\gamma_\alpha \leq \lambda$ and $\alpha < \lambda \Leftrightarrow \gamma_\alpha < \lambda$ [e.g., $\gamma_\alpha = \lambda^-(1 + \alpha)$ where $\lambda = (\lambda^-)^+$] and $\alpha < \beta \Rightarrow \gamma_\alpha < \gamma_\beta$.

(B) $\alpha < \beta \Rightarrow M_\alpha \subseteq M_\beta$.

(C) $\alpha < \beta$ & $\alpha \notin \mathcal{S} \Rightarrow M_\alpha$ is a direct summand of M_β.

(D) For limit $\delta \leq \lambda$, $M_\delta = \bigcup_{\alpha < \delta} M_\alpha$.

(E) M_0 is the zero bimodule.

(F) If α is successor ordinal or $\alpha \notin \mathcal{S}$: $M_{\alpha+1}$ is the direct sum of M_α and $\|M_\alpha\|$ copies of N_n, N_n' for each n and some others; each bimodule of power $< \lambda$ appears as a direct summand of $M_{\alpha+1}/M_\alpha$ for a stationary set of such α's.

(G) If $\alpha = \gamma_\alpha \in \mathcal{S}$, $\diamondsuit_{\mathcal{S}}$ gives us F_α, an endomorphism of M_α, as an R-module and there is P satisfying

$\otimes_P^\alpha$ [P is a bimodule of cardinality $< \lambda$ extending M_α such that:
(i) if $\beta < \alpha$, $\beta \notin \mathcal{S}$ then M_β is a direct summand of P,
(ii) F_α cannot be extended to an R-endomorphism of P.]

Then $M_{\alpha+1}$ satisfies $\otimes_{M_{\alpha+1}}^\alpha$.

Otherwise, act as in clause (F).

Note. There is no problem in carrying out the construction: for condition (C) we use "$\mathcal{S}$ does not reflect".

Now let $M_\lambda =: \bigcup_{\alpha < \lambda} M_\alpha$, so M_λ is a bimodule with universe λ.

2.7. Main Fact. Suppose $M_\lambda \xrightarrow{F} M_\lambda$ is an R-endomorphism of M_λ (i.e., endomorphism as an R-module). *Then* for some $\alpha < \lambda$, $\alpha \notin \mathcal{S}$ and $n(*) < \omega$, we have:

$(\mathrm{Pr})^{n(*)}_\alpha[F]$ *if* h is a homomorphism from $N_{n(*)}$ to M_λ (as bimodules), *then* for every $l < \omega$ we have:

$$(xh)F \in M_\alpha + \varphi_l(M_\lambda) + \mathrm{Rang}(h).$$

Proof of 2.7. Suppose that the conclusion fails. So for every $\alpha < \lambda$ and $n < \omega$ there is a counterexample $h_{\alpha,n} : N_n \to M_\lambda$ to $(\mathrm{Pr})^n_\alpha[F]$, the failure involving $l(\alpha, n) < \omega$. Now

$$C =: \{\delta < \lambda : F \text{ maps } M_\delta \text{ into } M_\delta,\ M_\delta \text{ has universe } \delta \text{ and, for every } \alpha < \delta,\ n < \omega, \text{ we have: } \mathrm{Rang}(h_{\alpha,n}) \subseteq M_\delta\}$$

is a club of λ.

So for some $\alpha \in \mathbb{S}$, α is an accumulation point of $C \setminus \mathbb{S}$ and $\diamondsuit_{\mathbb{S}}$ gives us, for α, $F_\alpha = F \restriction \alpha$ (remember $\{\delta < \lambda : \delta \notin \mathbb{S}, \mathrm{cf}\,\delta = \aleph_0\}$ is stationary).

We shall construct P satisfying $\bigotimes^\alpha_P$.

This suffices; why? By clause (G) of 2.6 we know that $\bigotimes^\alpha_{M_{\alpha+1}}$ holds; on the other hand there is β, $\alpha < \beta < \lambda$ such that F maps M_β into M_β, so (by condition (C) from 2.6) there is a projection F' from M_β onto $M_{\alpha+1}$ and $(F \restriction M_{\alpha+1}) \circ F'$ is an R-homomorphism from $M_{\alpha+1}$ to $M_{\alpha+1}$, contradicting $\bigotimes^\alpha_{M_{\alpha+1}}$.

Construction of P. Choose α_n such that

$$0 = \alpha_0 < \alpha_1 < \alpha_2 < \cdots,$$

$$\alpha_n \in C \setminus \mathbb{S} \qquad \text{for } n > 0,$$

$$\mathrm{Rang}(h_{\alpha_n,n}) \subseteq M_{\alpha_{n+1}},$$

$$\alpha = \bigcup_{n<\omega} \alpha_n.$$

For $n > 0$, as $\alpha_n \in C \setminus \mathbb{S}$ we know that M_{α_n} is a direct summand of $M_{\alpha_{n+1}}$, so let $M_{\alpha_{n+1}} = M_{\alpha_n} \oplus K_n$. Let $K_0 = M_{\alpha_1}$. So M_α is the direct sum of $\{K_n : n < \omega\}$. Let $P^0 = \prod_{n<\omega} K_n$; i.e., the set of elements of P^0 is $\{\langle z_n : n < \omega\rangle : z_n \in K_n\}$, addition and multiplication — coordinatewise, but we identify $\langle z_n : n < \omega\rangle$ with $\sum_{n<k} z_n$ if $\bigwedge_{n \ge k} z_n = 0$; so M_α is a sub-bimodule of P^0. For each $n > 0$ we know that (as $h_{\alpha_n,n}$ is a homomorphism from the bi-module N_n to the bi-module $M_{\alpha_{n+1}}$ and by the definition of N_n — see 2.5(a)):

(α) $a_l x h_{\alpha_n,n} = \sum_{i<k_n} b_{l,i}(y_i) h_{\alpha_n,n}$ for $l < m_n$,

(β) $x h_{\alpha_n,n} F \notin M_{\alpha_n} + \mathrm{Rang}(h_{\alpha_n,n}) + \varphi_{l(\alpha_n,n)}(M_{\alpha_{n+1}})$

[note: the first two summands are sub-bimodules; the third, not necessarily, but is an additive subgroup].

Let g_n^* be the projection from $M_{\alpha_{n+1}}$ onto K_n, so

$$g_n^* \restriction K_n = \text{identity}_{K_n}, \qquad g_n^* \restriction M_{\alpha_n} = \text{zero}$$

(note: g_n^* is a homomorphism of bimodules).

Clearly by (α):

$(\alpha)'$ $a_l x h_{\alpha_n,n} g_n^* = \sum_{i<k_n} b_{l,i} y_i h_{\alpha_n,n} g_n^*$ for $l < m_n$.

Now by the choice of g_n^*, as $\operatorname{Rang} h_{\alpha_n,n} \in M_{\alpha_{n+1}}$:

(γ) $xh_{\alpha_n,n} - xh_{\alpha_n,n} g_n^* \in M_{\alpha_n}$ and

(δ) $y_i h_{\alpha_n,n} - y_i h_{\alpha_n,n} g_n^* \in M_{\alpha_n}$,

(ϵ) $M_{\alpha_n} + \operatorname{Rang}(h_{\alpha_n,n}) = M_{\alpha_n} + \operatorname{Rang}(h_{\alpha_n,n} g_n^*)$,

hence clearly by (β) (and the choice of g_n^*):

(β') $xh_{\alpha_n,n} g_n^* \notin M_{\alpha_n} + \operatorname{Rang}(h_{\alpha_n,n} g_n^*) + \varphi_{l(\alpha_n,n)}(M_{\alpha_{n+1}})$.

Let $\mathcal{U} \subseteq \omega$ be infinite such that:

$$[n < m \ \& \ n \in \mathcal{U} \ \& \ m \in \mathcal{U} \Rightarrow l(\alpha_n, n) < m], \qquad 0 \notin \mathcal{U}.$$

We define x^n, y_i^n $(n, i < \omega)$:

$$\text{for } n \notin \mathcal{U}: \quad y_i^n = 0 \in K_n,$$
$$x^n = 0 \in K_n;$$
$$\text{for } n \in \mathcal{U}: \quad y_i^n = y_i h_{\alpha_n,n} g_n^* \quad \text{for } \underline{i < k_{m_n-1}},$$
$$y_i^n = 0 \qquad \text{for } \underline{i \geq k_{m_n-1}} \quad (\text{but } < \omega),$$
$$x^n = xh_{\alpha_n,n} g_n^*.$$

Now we define in P^0 some elements:

$$x^* = \langle x^n : n \leq \omega \rangle,$$
$$y_i^* = \langle y_i^n : n < \omega \rangle,$$
$$x^{*,j} = x^* - \sum_{n<j} x^n; \text{ i.e., } x^{*,j} = \langle \underbrace{0,0,\ldots,0}_{0,\ldots,j-1}, x^j, x^{j+1}, \ldots \rangle,$$
$$y_i^{*,j} = y_i^* - \sum_{n<j} y_i^n; \text{ i.e., } y_i^{*,j} = \langle \underbrace{0,0,\ldots,0}_{0,\ldots,j-1}, y_i^j, y_i^{j+1}, \ldots \rangle.$$

We can check that by $(\alpha)'$ [and for $n \notin \mathcal{U}$ trivially]:

$(\alpha)''$ $K_n \vDash [a_l x^n = \sum_{i<k_l} b_{l,i} y_i^n]$ when $l < m_n$;

hence

$(\alpha)'''$ $P^0 \models a_l x^{*,j} = \sum_{i<k_l} b_{l,i} y_i^{*,j}$ when $l < m_j$.

Now we define P:

P is the sub-bimodule of P^0 generated by $M_\alpha \cup \{x^*, y_i^* : i < \omega\}$.

Note that for $i,j < \omega$, $x^{*,j}, y_i^{*,j}$ belongs to P.

Suppose F^+ is an extension of $F_\alpha = F \restriction M_\alpha$ (which is an endomorphism of M_α as an R-module) to an endomorphism of P (as an R-module). Therefore $(x^*)F^+ \in P$, so for some $i(*) < \omega$, $\langle r_i : i < i(*)\rangle$ from R, $\langle s_i : i < i(*)\rangle$ from S:

(1) $x^*F^+ - \sum_{i<i(*)} r_i y_i^* s_i \in M_\alpha$ (remember $y_0^* = x^*$).

As $M_\alpha = \sum_{l<\omega} K_l$, for some $n(*) < \omega$ and some $z \in \sum_{l<n(*)} K_l = M_{\alpha_{n(*)}}$ we have

(2) $x^*F^+ - \sum_{i<i(*)} r_i y_i^* s_i = z$.

Without loss of generality $n(*) \in \mathfrak{U}$ (as we can increase $n(*)$, $\mathfrak{U} \subseteq \omega$ infinite). Let $m(*) = \operatorname{Min}[\mathfrak{U} \setminus (n(*) + 1)]$. We know that

(3) $x^{*,(n(*)+1)} = x^* - \sum\{x^n : n < n(*) + 1\} = x^* - \sum_{n<m(*)} x^n$ (as $n \notin \mathfrak{U} \Rightarrow x^n = 0$) satisfies $\varphi_{m(*)}(-)$ (in P!, by $(\alpha)'''$) hence also $x^{*,(n(*)+1)}F^+ = x^*F^+ - \sum\{x^nF : n < n(*) + 1\}$ satisfies it in P.

Let $Z_{n(*)}$ be the natural projection of P^0 onto $K_{n(*)}$: $(\langle v_0, v_1, v_2, \ldots, \rangle)Z_{n(*)} = v_{n(*)}$; so $Z_{n(*)}$ extends $g^*_{n(*)}$ and

(4) $x^{*,(n(*)+1)}(F^+Z_{n(*)}) = (x^*F^+)Z_{n(*)} - \sum\{(x^nF)Z_{n(*)} : n < n(*) + 1\}$.

The left-hand side satisfies $\varphi_{m(*)}(-)$ as an R-endomorphism preserves such satisfaction, hence also the right-hand side satisfies $\varphi_{m(*)}(-)$. Now for $n < n(*)$, $x^n \in M_{\alpha_{n+1}}$ hence (as $\alpha_{n+1} \in C$) $x^nF \in M_{\alpha_{n+1}} \subseteq M_{\alpha_{n(*)}} \subseteq \operatorname{Ker} Z_{n(*)}$, therefore $x^nFZ_{n(*)} = 0$. So the right-hand side of (4) is equal to $(x^*F^+)Z_{n(*)} - (x^{n(*)}F)Z_{n(*)}$. Now as $Z_{n(*)}$ extends $g^*_{n(*)}$ and $x^{n(*)}F \in M_{\alpha_{n(*)+1}}$, clearly

(5) $(x^{n(*)}F)Z_{n(*)} = (x^{n(*)}F)g^*_{n(*)}$.

So the right-hand side of the equation (5) is equal to $(x^*F^+)Z_{n(*)} - (x^{n(*)}F)g^*_{n(*)}$, hence (see line after (4) and remember Z is a homomorphism into $K_{n(*)}$):

(6) $K_{n(*)} \models \varphi_{m(*)}\left[(x^*F^+)Z_{n(*)} - (x^{n(*)}F)g^*_{n(*)}\right]$.

So

(7) $x^*F^+Z_{n(*)} - (x^{n(*)}F)g^*_{n(*)} \in \varphi_{m(*)}(K_{n(*)}) \subseteq \varphi_{m(*)}(M_{\alpha_{n(*)+1}})$.

By choice of $g^*_{n(*)}$ we have

(8) $x^{n(*)}F - (x^{n(*)}F)g^*_{n(*)} \in M_{\alpha_{n(*)}}$

and by the choice of $n(*)$ (and as $Z_{n(*)}$ is a homomorphism of bimodules and $z \in M_{\alpha_{n(*)}}$, hence $zF^+ = zF \in M_{\alpha_{n(*)}}$):

(9) $(x^*F^+)Z_{n(*)} = (x^*F^+ - 0)Z_{n(*)} = (x^*F^+ - z)Z_{n(*)} = \left(\sum_{i<i(*)} r_i y_i^* s_i\right)Z_{n(*)}$
$= \sum_{i<i(*)} r_i \left(y_i^* Z_{n(*)}\right)s_i = \sum_{i<i(*)} \left(r_i y_i^{n(*)}\right)s_i$
$= \sum_{i<i(*)} r_i y_i \left(h_{\alpha_{n(*)},n(*)} g^*_{n(*)}\right)s_i \in \operatorname{Rang}\left(h_{\alpha_{n(*)},n(*)} g^*_{n(*)}\right)$

[for the second equality note that $z \in M_{\alpha_{n(*)}}$ hence $zZ_{n(*)} = 0$ as $Z \restriction M_{\alpha_{n(*)}}$ is zero].

As $g^*_{n(*)}$ is a homomorphism with domain $M_{\alpha_{n(*)+1}}$ such that $(\forall y \in M_{\alpha_{n(*)+1}})$ $[y - yg^*_{n(*)} \in M_{\alpha_{n(*)}}]$ we have (remember: $x \in N_{n(*)}$ and $x^n = xh_{\alpha_{n(*)},n(*)}g^*_{n(*)}$ — see choice of the x^n's):

(10) $xh_{\alpha_{n(*)},n(*)}F - xh_{\alpha_{n(*)},n(*)}Fg^*_{n(*)} \in M_{\alpha_{n(*)}}$

and (as F maps $M_{\alpha_{n(*)}}$ into itself)

(11) $xh_{\alpha_{n(*)},n(*)}Fg^*_{n(*)} - xh_{\alpha_{n(*)},n(*)}g^*_{n(*)}F \in M_{\alpha_{n(*)}}$,

and by the choice of the x^n's

(12) $x^{n(*)} = xh_{\alpha_{n(*)},n(*)}g^*_{n(*)}$; hence

$x^{n(*)}F = xh_{\alpha_{n(*)},n(*)}g^*_{n(*)}F.$

By the last equations [first (10), (11), (12), then (8) and then (7) + (9)]:

$$\begin{aligned} xh_{\alpha_{n(*)},n(*)}F &\in (x^{n(*)})F + M_{\alpha_{n(*)}} = (x^{n(*)}F)g^*_{n(*)} \\ &\subseteq M_{\alpha_{n(*)}} + \mathrm{Rang}\big(h_{\alpha_{n(*)},n(*)}\big) + \varphi_{m(*)}\big(M_{\alpha_{n(*)+1}}\big) \end{aligned}$$

so we get a contradiction to the choice of $h_{\alpha_{n(*)},n(*)}$.

Hence we have proved 2.7.

2.8. Definition. (1) $\mathrm{HDS}^{M_2}_{M_1}(h,N)$ means: M_1, M_2, N are bimodules, $M_1 \subseteq M_2$, h a (bimodule) homomorphism from N into M_2 and, for some bimodule K, $M_2 = M_1 \oplus (\mathrm{Rang}\, h) \oplus K$.

(2) $\mathrm{IDS}^{M_2}_{M_1}(h,N)$ is defined similarly but h is one to one.

(3) $(\mathrm{Pr}^-)^{n(*)}_\alpha[F]$ is the following apparent weakening of $(\mathrm{Pr})^{n(*)}_\alpha[F]$ (speaking on $\langle M_\alpha : \alpha \le \lambda \rangle$):

if $\mathrm{IDS}^{M_\beta}_{M_\alpha}(h, N_{n(*)})$, $\alpha < \beta < \lambda$, $\beta \notin \mathbb{S}$
then for each $l < \omega$ we have:

$$(xh)F \in M_\alpha + (\mathrm{Rang}\, h) + \varphi_l(M_\lambda).$$

2.9. Fact. (1) If $\mathrm{IDS}^{M_2}_{M_1}(h_1,N)$ and h_0 is a bimodule homomorphism from N into M_1, and $h =: h_0 + h_1$, then $\mathrm{IDS}^{M_2}_{M_1}(h,N)$.

(2) If $M_0 \subseteq M_1 \subseteq M_2$ are bimodules, M_0 a direct summand of M_1, $\mathrm{IDS}^{M_2}_{M_1}(h,N)$ *then* $\mathrm{IDS}^{M_2}_{M_0}(h,N)$.

(3) If $(\mathrm{Pr}^-)^{n(*)}_\alpha[F]$, $\alpha \le \beta < \lambda$, F maps M_α into itself, $\alpha \notin \mathbb{S}$, $\beta \notin \mathbb{S}$ then $(\mathrm{Pr}^-)^{n(*)}_\beta[F]$.

(4) If $(\mathrm{Pr})^{n(*)}_\alpha[F]$ then $(\mathrm{Pr}^-)^{n(*)}_\alpha[F]$.

Proof. Direct checking.

2.10. CLAIM. Suppose $\langle M_\alpha : \alpha \leq \lambda \rangle$ and $\mathcal{S}$ satisfy (A)–(F) of 2.6 (but not necessarily (G)!) and $F: M_\lambda \to M_\lambda$ is an endomorphism of M_λ as an R-module and $(\mathrm{Pr}^-)^{n(*)}_\alpha[F]$ holds (see 2.8(3)) and $\alpha \notin \mathcal{S}$.

Then for some $z \in N^{\mathrm{tr}}_{n(*)}$ (on $N^{\mathrm{tr}}_{n(*)}$ see 2.5(f)) we have:

$(\mathrm{Pr1})^{n(*)}_{\alpha,z}[F]$ if h is a homomorphism from $N_{n(*)}$ to M_λ then $(xh)F - zh \in M_\alpha + \bigcap_{l<\omega} \varphi_l(M_\lambda)$.

PROOF OF 2.10.

Step a. We shall prove: if $\mathrm{IDS}^{M_\beta}_{M_\alpha}(h, N_{n(*)})$ for some $\beta \in (\alpha, \lambda) \setminus \mathcal{S}$ then $xhF \in M_\alpha + h(N) + \bigcap_l \varphi_l(M_\lambda)$.

Assume $\alpha < \beta \notin \mathcal{S}$, $M_\beta = M_\alpha \oplus N \oplus K$ (bimodules direct sum), h an isomorphism from $N_{n(*)}$ onto N (i.e., $\mathrm{IDS}^{M_\beta}_{M_\alpha}(h, N_{n(*)})$. Choose $\gamma > \beta$ such that F maps M_γ into itself and $\gamma \notin \mathcal{S}$, so M_β is a direct summand of M_γ hence $M_\gamma = M_\alpha \oplus N \oplus K'$. Let Z be the projection from M_γ onto K' with kernel $M_\alpha \oplus N$ (as bimodules); we know that for each l

$$(xh)F \in M_\alpha + N + \varphi_l(M_\gamma).$$

Clearly for some $v \in M_\alpha$, $u \in N$ and $w \in \varphi_l(M_\gamma)$ we have $xhF = v + u + w$, hence

$$xhFZ = vZ + uZ + wZ = 0 + 0 + wZ = wZ$$

so

$$xhFZ \in (\varphi_l(M_\gamma))Z \subseteq \varphi_l(M_\gamma).$$

As this holds for each l

$$xhFZ \in \bigcap_l \varphi_l(M_\gamma) \subseteq \bigcap_l \varphi_l(M_\lambda).$$

So $(xh)F = [(xh)F - ((xh)F)Z] + (xhF)Z \in (M_\alpha \oplus N) + \bigcap_{l<\omega} \varphi_l(M_\lambda) = M_\alpha + (\mathrm{Rang}\, h) + \bigcap_{l<\omega} \varphi_l(M_\lambda)$.

Step b. Assume that for $\zeta = 1,2$, $\alpha < \beta_\zeta \notin \mathcal{S}$, $\beta_\zeta < \lambda$, $M_{\beta_\zeta} = M_\alpha \oplus N^*_\zeta \oplus K_\zeta$ (bimodule direct sum), h_ζ is an isomorphism from $N_{n(*)}$ onto N^*_ζ, $z_\zeta \in N_{n(*)}$ such that $[xh_\zeta F - z_\zeta h_\zeta \in M_\alpha + \bigcap_{l<\omega} \varphi_l(M_\lambda)]$. *Then* (in $N_{n(*)}$):

$$z_1 \equiv z_2 \bmod \bigcap_{l<\omega} \varphi_l(N_{n(*)}).$$

We choose $\beta \notin S$, $\beta > \beta_1$, $\beta > \beta_2$, $\beta < \lambda$ such that F maps M_β into M_β. Let N_3^* be isomorphic to $N_{n(*)}$ such that $M_{\beta+1}$ is the direct sum of M_β, N_3^* and some others (just remember (F) of 2.6).

Let h_3 be an isomorphism from $N_{n(*)}$ onto N_3^* and $z_3 \in N_{n(*)}$ be such that

$$xh_3F - z_3h_3 \in M_\alpha + \bigcap_{l<\omega} \varphi_l(M_\lambda)$$

(exists by stage a).

It is enough to prove $z_3 \equiv z_1$ and $z_3 \equiv z_2 \bmod[\bigcap_l \varphi_l(N_{n(*)})]$ in $N_{n(*)}$; and by symmetry it is enough to prove $z_3 \equiv z_1$. Clearly for some K, $M_{\beta+1} = M_\alpha \oplus N_1^* \oplus N_3^* \oplus K$. Let $N_4^* = \{vh_1 - vh_3 : v \in N_{n(*)}\}$ and define $h_4 : N_{n(*)} \to M_{\beta+1}$ by

$$vh_4 = vh_1 - vh_3.$$

Clearly N_4^* is a sub-bimodule of M_λ, h_4 an isomorphism from $N_{n(*)}$ onto N_4^* and $M_{\gamma+1} = M_\alpha \oplus N_1^* \oplus N_4^* \oplus K$. Now modulo $M_\alpha + \bigcap_{l<\omega} \varphi_l(M_\lambda)$:

$$(*) \qquad (xh_4)F = (xh_1 - xh_3)F = xh_1F - xh_3F \equiv z_1h_1 - z_3h_3.$$

Now by step a:

$$(*)_1 \qquad (xh_4)F \in \mathrm{Rang}(h_4) + \left(M_\alpha + \bigcap_{l<\omega} \varphi_l(M_\lambda)\right).$$

So

$$(*)_2 \qquad z_1h_1 - z_3h_3 \in \mathrm{Rang}\, h_4 + \left(M_\alpha + \bigcap_{l<\omega} \varphi_l(M_\lambda)\right).$$

By $(*)_2$ and the definitions of h_4, for some $v \in N_{n(*)}$,

$$(z_1h_1 - z_3h_3) - (vh_1 - vh_3) \in M_\alpha + \bigcap_{l<\omega} \varphi_l(M_\lambda);$$

i.e., $(z_1 - v)h_1 - (z_3 - v)h_3 \in M_\alpha + \bigcap_{l<\omega} \varphi_l(M_\lambda)$. So for some $y \in M_\alpha$ we have $(z_1 - v)h_1 - (z_3 - v)h_3 - y \in \bigcap_{l<\omega} \varphi_l(M_\lambda)$.

But $M_{\gamma+1} = M_\alpha \oplus N_1^* \oplus N_3^* \oplus K$ and $\bigcap_{l<\omega} \varphi_l(M_\lambda) \cap M_{\gamma+1} = \bigcap_{l<\omega} \varphi_l(M_{\gamma+1})$, so as $(z_1 - v)h_1 - (z_3 - v)h_3 \in N_1^* \oplus N_3^*$, without loss of generality $y = 0$. Also

$$\begin{aligned}\bigcap_{l<\omega} \varphi_l(M_\lambda) \cap (N_1^* \oplus N_3^*) &= \bigcap_{l<\omega} \varphi_l(N_1^* \oplus N_3^*) \\ &= h_1''\left(\bigcap_{l<\omega} \varphi_l(N_{n(*)})\right) + h_3''\left(\bigcap_{l<\omega} \varphi_l(N_{n(*)})\right);\end{aligned}$$

we have

$$(z_3 - v)h_1 - (z_1 - v)h_3 \in h_1''\left(\bigcap_{l<\omega} \varphi_l(N_{n(*)})\right) + h_3''\left(\bigcap_{l<\omega} \varphi_l(N_{n(*)})\right).$$

Now in $N_1^* \oplus N_3^*$ this implies for $\zeta = 1,3$

$$(z_\zeta - v)h_\zeta \in h_\zeta''\left(\bigcap_l \varphi_l(N_{n(*)})\right);$$

i.e., $z_\zeta - v \in \bigcap_l \varphi_l(N_{n(*)})$. Hence also (in $N_{n(*)}$)

$$z_1 - z_3 = (z_1 - v) - (z_3 - v) \in \bigcap_{l<\omega} \varphi_l(N_{n(*)}).$$

So $z_1 - z_3 \in \bigcap_l \varphi_l(N_{n(*)})$; i.e., we finish step b.

Step c. There is $z \in N_{n(*)}$ such that, if h is a homomorphism from $N_{n(*)}$ into M_λ, *then*

$$xhF - zh \in M_\alpha + \bigcap_l \varphi_l(M_\lambda),$$

By stage b there is $z \in N_{n(*)}$ which satisfies the above requirement when h is as there. Suppose h_0 is a counterexample. Choose $\beta \notin \mathcal{S}$, $\beta > \alpha$, F maps M_β into M_β and $\mathrm{Rang}(h_0) \subseteq M_\beta$. Let h_1 be an isomorphism from $N_{n(*)}$ onto some N_1^* such that $M_{\beta+1} = M_\beta \oplus N_1^* \oplus K$ for some K. So

$$xh_1F - zh_1 \in M_\alpha + \bigcap_{l<\omega} \varphi_l(M_\lambda).$$

Let $N_{n(*)} \xrightarrow{h_2} M_\lambda$ be defined by

$$vh_2 = vh_1 - vh_0.$$

Easily h_2 is a bimodule homomorphism and, by the assumptions on N_1^*, h_1 (direct sum isomorphism), h_2 is an isomorphism from $N_{n(*)}$ onto $N_2^* =: \mathrm{Rang}(h_2)$, and

$$M_{\beta+1} = M_\beta \oplus N_2^* \oplus K.$$

So by step b, $xh_2F - zh_2 \in M_\alpha + \bigcap_l \varphi_l(M_\lambda)$. But

$$(xh_0)F = (xh_1 - xh_2)F = xh_1F - xh_2F \in zh_1 - zh_2 + \left(M_\alpha + \bigcap_{n<\omega} \varphi_n(M_\lambda)\right)$$

$$= zh_0 + \left(M_\alpha + \bigcap_{n<\omega} \varphi_n(M_\lambda)\right)$$

as required, so we have proved z as required exists.

Step d. $z \in N^{tr}_{n(*)}$ (z from step c). ($N^{tr}_{n(*)}$ is defined in (f) of 2.5.)

PROOF. $z \in \varphi_{n(*)}(N_{n(*)})$ is very easy.

Let $h : N'_{n(*)} \to N^*_1 \subseteq M_{\alpha+1}$ be an isomorphism (onto) such that, for some sub-bimodule K, $M_{\alpha+1} = M_\alpha \oplus N^*_1 \oplus K$ [see 2.5(e) for definition of $N'_{n(*)}, f^\xi_{n(*)}$ and condition (F) of 2.6]. So

$$N_{n(*)} \xrightarrow{f^1_{n(*)}h} M_\lambda, \qquad N_{n(*)} \xrightarrow{f^2_{n(*)}h} M_\lambda$$

are homomorphisms, so for $\zeta = 1,2$

$$(x(f^\zeta_{n(*)}h))F - z(f^\zeta_{n(*)}h) \in M_\alpha + \bigcap_{l<\omega} \varphi_l(M_\lambda)$$

and the conclusion follows.

2.11. DISCUSSION. (a) Now $(\mathrm{Pr1})^{n(*)}_{\alpha,z}[F]$ (from 2.10) is almost what is required, only the "error term" M_α is too large.

(b) However, before we do this, we note that for the solution of the Kaplansky test problem this improvement is immaterial: we just divide by a stronger ideal, i.e., we allow one to divide by a submodule of bigger cardinality. We phrase our conclusion more clearly before we proceed.

2.12. DEFINITION. (1) For any $n < \omega$, $z \in N^{tr}_n$ and bi-module M, we define $H^z_M = {}^nH^z_M$.

H^z_M is the function from the abelian group $\varphi_n(M)/\bigcap_{l<\omega}\varphi_l(M)$ to itself defined by:

if h is a homomorphism from N_n to M, then

$$\left(xh + \bigcap_l \varphi_l(M)\right)H^z_M = zh + \bigcap_{l<\omega}\varphi_l(M).$$

(2) z is called n-nice if ($z \in N^{tr}_n$ and), when $h : N_n \to M$ is a homomorphism, $m > n$, $M \models \varphi_m(xh)$, then $M \models \varphi_m(zh)$.

2.13. CLAIM. (1) For n, z, M as in 2.12, ${}^nH^z_M$ is really a single-valued function and an endomorphism of the abelian group $\varphi_n(M)/\bigcap_{l<\omega}\varphi_l(M)$, so the value depends just on $z + \bigcap_l \varphi_l(N_n)$. Also if $z_1, z_2 \in N^{tr}_n$, $z_1 - z_2 \notin \bigcap_{l<\omega}\varphi_l(N_n) \Rightarrow$ for some R-module M, ${}^nH^{z_1}_M \neq {}^nH^{z_2}_M$ (e.g., $M = N_n$).

(2) If M_1, M_2 are R-modules, $h : M_1 \to M_2$ a homomorphism, then:

(i) $(\varphi_l(M_1))h \subseteq \varphi_l(M_2)$.

(ii) For $n < \omega$, we define $\hat{h}$: for $x \in \varphi_n(M)$ we let

$$\left(x + \bigcap_{l<\omega}\varphi_l(M_1)\right)\hat{h} =: xh + \bigcap_{l<\omega}\varphi_l(M_2),$$

$\hat{h}$ is a homomorphism from $\varphi_n(M_1)/\bigcap_l \varphi_l(M_1)$ into $\varphi_n(M_2)/\bigcap_l \varphi_l(M_2)$ (as abelian groups). We denote $\hat{h}$ by $h \restriction \varphi_n(M_1)/\bigcap_{l<\omega} \varphi_l(M_1)$.

(iii) If $n < \omega$, $z \in N_n^{\mathrm{tr}}$, M_1 and M_2 are bi-modules, then

$$^nH^z_{M_1} \circ \hat{h} = \hat{h} \circ {}^nH^z_{M_2}.$$

(3) If $n < m$, $z \in N_n^{\mathrm{tr}}$ is n-nice, then for some $y \in N_m^{\mathrm{tr}}$ for every bi-module M:

$$^mH^y_M = {}^nH^z_M \restriction \left(\varphi_m(M) \Big/ \bigcap_{l<\omega} \varphi_l(M)\right).$$

(4) Suppose:

(i) $\psi(x,y)$ is a p.e. formula in the language of bi-modules, logic $-\mathcal{L}_{\lambda,\omega}$.

(ii) $\varphi_n(x) \to (\exists y)\psi(x,y)$, i.e., this holds for every x in every bimodules.

(iii) $\psi(x,y) \to \varphi_n(x)$ & $\varphi_n(y)$ (i.e., as in (ii)).

(iv) $\psi(x,y_1)$ & $\psi(x,y_1) \to \varphi_l(y_1 - y_2)$ for $l < \omega$ (i.e., as in (ii)).

Then for some $z \in N_n^{\mathrm{tr}}$:

$(*)^n_{\psi,z}$ for every bimodule M:

$$\left\{\left\langle x + \bigcap_l \varphi_l(M), y + \bigcap_l \varphi_l(M)\right\rangle : M \models \psi[x,y]\right\}$$

$$= \left\{\left\langle x + \bigcap_l \varphi_l(M), y + \bigcap_l \varphi_l(M)\right\rangle : \left(x + \bigcap_l \varphi_l(M)\right) H^z_M = y + \bigcap_l \varphi_l(M) \right.$$

$$\left. (\text{so } x,y \in \varphi_n(M))\right\}.$$

(5) For every $z \in N_n^{\mathrm{tr}}$ for some $\psi(x,y)$, (i), (ii), (iii), (iv) and $(*)^n_{\psi,z}$ holds. (In fact, the formula is first order conjunctive positive existential.)

(6) For every $n < \omega$ and $z_1, z_2 \in N_n^{\mathrm{tr}}$ for some $z_3 \in N_n^{\mathrm{tr}}$: for every M, $^nH^{z_3}_M = {}^nH^{z_1}_M \circ {}^nH^{z_2}_M$; and $z_4 = z_1 \neq z_2$ is in N_n^{tr} and satisfies, for every R-module M, $^nH^{z_4}_M = {}^nH^{z_1}_M \circ -{}^nH^{z_2}_M$.

(7) If $z \in N_n^{\mathrm{tr}}$ and $^nH^z_{N_n}$ is one to one and onto (i.e., from $\varphi_n(N_n)/\bigcap_l \varphi_l(N_n)$ onto itself) *then* for some $z' \in N_n^{\mathrm{tr}}$ for every R-module M, $^nH^{z'}_M$ is the inverse of $^nH^z_M$.

(8) In (4), (5), (6), (7) we can start with $S = T = \mathrm{Cent}\, R$ so ψ is the language of R-modules, and the parallel result holds.

PROOF. Left to the reader. [For (6) and for (7) use (5) and then (4).]

2.14. DEFINITION. For an R-module M let:

(1) $\mathrm{End}(M)$ = ring of endomorphisms of M.

$\mathrm{End}^{\bar\varphi,n}(M) = \{[h \restriction \varphi_n(M)]/\bigcap_l \varphi_l(M) : h \in \mathrm{End}(M)\}$.

$\mathrm{End}^{\bar\varphi,n}_{<\lambda}(M) = \{[h \upharpoonright \varphi_n(M)]/\bigcap_{l<\omega}\varphi_l(M) \in \mathrm{End}^{\bar\varphi,n}(M)$: for some $A \subseteq M$, $|A| < \lambda$
and $\mathrm{Rang}\,\hat h \subseteq \{x + \bigcap_l \varphi_l(M) : x \in \varphi_n(\langle A\rangle_M)\}\}$.

$\mathrm{End}^{\bar\varphi,\omega}_{(<\lambda)}(M)$ is the direct limit of $\langle \mathrm{End}^{\bar\varphi,n}_{(<\lambda)}(M) : n < \omega\rangle$ with the natural mappings $\Phi^{n,m}_{(<\lambda)}[M]$ from $\mathrm{End}^{\bar\varphi,n}_{(<\lambda)}(M)$ to $\mathrm{End}^{\bar\varphi,m}_{(<\lambda)}(M)$.

(2) $B^n_{\bar\varphi}(M)$ is $\varphi_n(M)/\bigcap_l\varphi_l(M)$ expanded by the finitary relations definable by p.e. formulas (say in $\mathfrak{L} = \mathfrak{L}_{(2^{|R|+|S|+\aleph_0})^+,\omega}$) in ${}_RM$ (so actually even if we use this for a bimodule M, it counts only as an R-module).

(3) ${}^+B^n_{\bar\varphi}(M)$ is defined similarly, but p.e. is replaced by: preserved by direct sums.

2.15. Fact. (1) In 2.14(1) all are rings into which (if M is a bimodule) S is mapped naturally†; $\mathrm{End}^{\bar\varphi,n}_{<\lambda}$ is a two-sided ideal of $\mathrm{End}^{\bar\varphi,n}_{<\mu}$ if $\lambda < \mu$, $\mathrm{End}^{\bar\varphi,n}_{<|M|^+}(M) = \mathrm{End}^{\bar\varphi,n}(M)$.

(2) If M_1, M_2 are R-modules, h a homomorphism from M_1 to M_2 as R-module, then h induces a homomorphism from $B^n_{\bar\varphi}(M_1)$ into $B^n_{\bar\varphi}(M_2)$ naturally.

(3) For a bimodule $M, z \in N^{\mathrm{tr}}_n$, the function ${}^nH^z_M$ is definable by a p.e. formula (this is 2.13(5)). If (in N_n) $z \in \sum_{i<k_{m_n-1}} Ry_i$, the p.e. formula is in the language of R-modules.

The rings $dE^n(dE)$ defined below are derived from the ring of R-endomorphisms of bimodules which we have not discarded. Note 2.13.

2.16. Definition. (1) Let DE^n be the following ring; its elements are the (formal) operators ${}^nH^z$ for $z \in N^{\mathrm{tr}}_n$:

(a) ${}^nH^{z_1} = {}^nH^{z_2}$ iff $z_1 - z_2 \in \bigcap_l \varphi_l(N_n)$.

(b) ${}^nH^{z_1} \pm {}^nH^{z_2} = {}^nH^{z_1 \pm z_2}$.

(c) ${}^nH^{z_1} \circ {}^nH^{z_2} = {}^nH^{z_3}$, if for each bimodule this holds (z_3 exists, by 2.13(6); unique (mod $\bigcap_l\varphi_l(N_n)$), by 2.13(1)).

(d) The zero is ${}^nH^0$, the one is ${}^nH^x$ (DE^n is a ring — as it is embedded into the endomorphism ring of the $\varphi_n(N_n)/\bigcap_l\varphi_l(N_n)$ as an abelian group).

(2) $De^n = \{{}^nH^z \in DE^n : z \in \sum_i Ry_i\}$ is a subring of DE^n.

(3) $dE^n = \{{}^nH^z \in DE^n : {}^nH^z_M$ is an endomorphism of $B^n_{\bar\varphi}(M)$ for every bimodule $M\}$.

$dE^n_1 = \{{}^nH^z : z \in N^{\mathrm{tr}}_n$ and z is n-nice$\}$.

(4) $de^n =: De^n \cap dE^n$, $de^n_1 \stackrel{\mathrm{def}}{=} De^n \cap dE^n_1$.

(5) $de^n(R)$ is de^n when we choose $S = T = \mathrm{Cent}(R)$; similarly for the others.

†For each $s \in S$, M a bimodule, s defines an endomorphism of M as an R-module: $x \mapsto xs$; now apply 2.13(4). Is it an embedding? Not necessarily, e.g. if $\varphi_n(x)$ is "x divisible by z^n", if $s = 2^ns_n \in S$ for each n, then s is mapped to zero.

2.17. Claim. (1) DE^n is a ring, De^n, dE^n subrings, dE_1^n is a subring of DE^n extending dE^n (all have the unit $1 = {}^nH^x$ and zero ${}^nH^0$, and extending T).

(2) De^n, dE^n commute, hence de^n is commutative.

(3) There is a natural homomorphism from dE^n to dE^{n+1} $(n < \omega)$, the direct limit is denoted by dE. Similarly for dE_1^n, dE_1. Also S is naturally mapped into dE^n which is naturally embedded (i.e., by the identity map) into dE_1^n; the diagram commutes. Similarly de^n is naturally embedded into de_1^n.

(4) $\varphi_n(M)/\bigcap_l \varphi_l(M)$ is naturally a module over DE^n and it is naturally a (De^n, dE^n)-bimodule (with de^n playing the role of T).

The following lemma says that, e.g., in the module we constructed in 2.7 (see 2.10) we have some control over $\mathrm{End}(M_\lambda)$; note that it only says it is not too large, but we have the freedom to choose the ring S in order to make $\mathrm{End}(M_\lambda)$ have some elements with desirable properties.

2.18. Lemma. *Suppose* $\langle M_\alpha : \alpha \le \lambda \rangle$ *satisfies* (A)–(F) *of* 2.6, $M = M_\lambda$ *and*

(*) *for every endomorphism* $F: M_\lambda \to M_\lambda$ *for some* $n < \omega$, $z \in N_n^{\mathrm{tr}}$, $\alpha \in \lambda \setminus \mathbb{S}$ *we have* $(\mathrm{Pr1})_{\alpha,z}^n[F]$.

Then:

(i) *If* $(\mathrm{Pr1})_{\alpha,z}^n[F]$ *then* ${}^nH_M^z$ *is an endomorphism of* $B_\varphi^n(M)$. *So as each* N_n *is isomorphic to a direct summand of* M_β *complimentary to* M_α *for* $\alpha < \beta$ *in* $\lambda \setminus \mathbb{S}$, z *is n-nice; i.e.* ${}^nH^z \in dE_1^n$. *Also as, e.g., "every* $\varphi(\bar{x})$, *a p.e. formula in* $\mathfrak{L}$ *which has a model, has a model which is a direct summand of* M", *clearly necessarily* ${}^nH^z \in dE^n$.

(ii) *If* $(\mathrm{Pr1})_{\alpha,z}^n[F]$ *and* F *is an automorphism of* M *then* ${}^nH_M^z$ *is an automorphism of* $B_\varphi^n(M)$ *and even of* ${}^+B_\varphi^n(M)$ [*we can use* 2.13(7)].

(iii) $\mathrm{End}^{\bar\varphi,\omega}(M_\lambda)/\mathrm{End}^{\bar\varphi,\omega}_{<\lambda}(M_\lambda)$ *can be embedded into the ring* dE (*see* 2.15, 2.16(3)); *moreover for every subring* $\mathfrak{C}$ *of* $\mathrm{End}^{\bar\varphi,\omega}(M_\lambda)/\mathrm{End}^{\bar\varphi,\omega}_{<\lambda}(M_\lambda)$ *of power* $< \lambda$, *for some club* C *of* λ, *if* $\alpha \in C \setminus \mathbb{S}$ *is large enough, then* $\mathfrak{C}$ *is embedded into* $\mathrm{End}^{\bar\varphi,\omega}(M_\lambda/M_\alpha)$

(iv) *Moreover*, $\mathrm{End}^{\bar\varphi,\omega}(M_\lambda) = \bigcup_{n<\omega} E_n$, $E_n \subseteq E_{n+1}$,

$$E_n = \{\Phi^{n,\omega}(F \restriction \varphi_n/\textstyle\bigcap_l \varphi_l) : F \in \mathrm{End}(M), \text{ and there are } z_n(F) \in N_n^{\mathrm{tr}},$$

$$\alpha_n(F) < \lambda \text{ such that } (\mathrm{Pr1})^n_{\alpha_n(F), z_n(F)}(F)\},$$

let $n(F) = \mathrm{Min}\{n \cdot F \in E_n\}$;

$$z_n(F) \text{ is unique modulo } \bigcap_{l<\omega} \varphi_l(N_n).$$

(v) E_n *is a subring of* $\mathrm{End}^{\bar\varphi,\omega}(M)$ *and the mapping* $F \mapsto {}^nH^{z_n(F)}$ *is a homomorphism from*

$$\left\{ F \restriction \varphi_n \Big/ \bigcap_l \varphi_l : F \in \mathrm{End}(M) \textit{ and } (\mathrm{Pr1})^n_{\alpha_n(F), z_n(F)} \right.$$
$$\left. \textit{for some } \alpha_n(F) < \lambda,\ z_n(F) \in N^{\mathrm{tr}}_n \right\}$$

into dE^n *with kernel* $\mathrm{End}^{\bar\varphi, n}_{<\lambda}(M)$; *i.e.* $\{F \in \mathrm{End}^{\bar\varphi, n}(M) : z_n(F) \in \bigcap_l \varphi_l(N_n)\}$.

(vi) *The ring S is naturally mapped into* $\mathrm{End}_R(M_\lambda)$, *for each $\alpha \le \omega$, there is a natural homomorphism from* $\mathrm{End}_R(M_\lambda)$ *to* $\mathrm{End}^{\bar\varphi, \alpha}(M_\lambda)$ *which, for $\alpha < \omega$, has a natural mapping to dE. (So S is naturally mapped into dE.)*

§3. Reducing the error term

3.1. Revised Context. (1) Let $g_n : N_n \to N_{n+1}$ be the homomorphism with $x^{[n]}g = x^{[n+1]}$, $y_i^{[n]}g = y_i^{[n+1]}$ for $i < k_{m_n - 1}$. Let $g_{n,m} = g_n g_{n+1} \cdots g_{m+1}$ for $n \le m < \omega$.

(2) Let $\mathcal{K}$ be a family of bimodules, each of power $< \lambda$, and $\mathcal{K}$ has $\le \lambda$ members, and $N_n, N'_n \in \mathcal{K}$ for each $n < \omega$. We call $\mathcal{K}$ trivial if $\mathcal{K} = \{N_n, N'_n : n < \omega\}$. Let $\mathrm{cl}_{\mathrm{is}}(\mathcal{K})$ be the class of bimodules isomorphic to some $K \in \mathcal{K}$. Let $\mathrm{cl}(\mathcal{K}) = \mathrm{cl}_{\mathrm{ds}}(\mathcal{K})$ be the class of bimodules isomorphic to a direct sum of bimodules from $\mathrm{cl}_{\mathrm{is}}(\mathcal{K})$ (so $\mathrm{cl}_{\mathrm{is}}(\mathrm{cl}(\mathcal{K})) = \mathrm{cl}(\mathcal{K})$). A $\mathcal{K}$-bimodule means a bimodule from $\mathrm{cl}_{\mathrm{is}}(\mathcal{K})$. We say M_1 is a $\mathcal{K}$-direct summand of M_2 if $M_2 = M_1 \oplus K$, $K \in \mathrm{cl}(\mathcal{K})$.

(3) We now redo §2. A bimodule of cardinality $<\lambda$ is usually replaced by a $\mathrm{cl}(\mathcal{K})$-bimodule. In particular, in 2.6:

In (A), $M_\alpha \in \mathrm{cl}(\mathcal{K})$ for $\alpha < \lambda$.

In (C), M_α is a $\mathrm{cl}(\mathcal{K})$-direct summand of M_β.

In (F), the other bimodules are from $\mathcal{K}$, and "each bimodule" is replaced by "each bimodule from $\mathcal{K}$" (so we have $\le \lambda$ assignments).

In Definition 2.8(1), $K \in \mathrm{cl}(\mathcal{K})$.

In 2.9(2), M_0 is a $\mathrm{cl}(\mathcal{K})$-direct summand of M_1.

In the proof of 2.10: check no harm is done.

In 2.16(3), "for every $\mathcal{K}$-bimodule".

In 2.18(i), ${}^nH^z \in dE^n$ remains; ${}^nH^z \in dE^n$ = we use the new definition of dE^n.

3.2. Claim. For any unbounded $\mathcal{U} \subseteq \omega$, letting $i(n) = i_{\mathcal{U}}(n)$ = the nth member of $\mathcal{U}$, there are bimodules $P_{\mathcal{U}}$, $P_{\mathcal{U},n}$ and $h^*_n : N_{i(n)} \to P_{\mathcal{U}}$ embeddings for $n < \omega$ and $x \in P_{\mathcal{U}}$ such that:

(a) $\mathrm{Rang}\, h^*_n \cap \sum_{m \ne n} \mathrm{Rang}\, h^*_m = \{0\}$.

(b) For each $n < \omega$ we have: $P_{\mathcal{U}} = \left(\sum_{l<n} \mathrm{Rang}\, h^*_l\right) \oplus K_n, K_n$ is a direct sum of copies of N_m's (and really of $N_{i(l)}, l \ge n$); let $P_{\mathcal{U},n} =: \sum_{l<n} \mathrm{Rang}\, h^*_l$.

(c) $\sum_{n<\omega} \operatorname{Rang} h_n^*$ is not a direct summand of $P_{\mathfrak{U}}$; moreover, there are $x \in P_{\mathfrak{U}}$, $x \notin \sum_{n<\omega} \operatorname{Rang} h_n^* + \bigcap_n \varphi_n(P_{\mathfrak{U}})$ and $f: N_{i(0)} \to P_{\mathfrak{U}}$ a homomorphism, $x^{[i(0)]}f = x$, such that, for each n for some

$$x_n =: \sum_{l<n} (x^{[i(l)]})h_l^* \in \sum_{l<n} \operatorname{Rang} h_l^*,$$

$$x - x_n \in \varphi_{i(n)}(P_{\mathfrak{U}}) \quad \text{and} \quad (x^{[i(0)]})f = x,$$

$$P_{\mathfrak{U}} = \left\langle \bigcup_n \operatorname{Rang} h_n^* \cup \operatorname{Rang} f \right\rangle.$$

(d) $P_{\mathfrak{U}}$ is the direct sum of copies of the N_n's.

Proof. Let $P_{\mathfrak{U}}$ be $\bigoplus_{i<\omega} \operatorname{Rang} f_i^*$, $f_n^* : N_{i(n)} \to P_{\mathfrak{U}}$ an embedding, $i(n)$ the nth member of $\mathfrak{U}$ (i.e., $P_{\mathfrak{U}}$ is the direct sum of the N_n's for $n \in \mathfrak{U}$ so (d) holds). We define $h_n^* : N_{i(n)} \to M$ by induction on n (on $g_{n,n+1}$, see 3.1(1)):

$$th_n^* =: tf_n^* - tg_{i(n),i(n+1)} f_{n+1}^*.$$

Clearly h_n^* is a homomorphism. As $P_{\mathfrak{U}} = \operatorname{Rang} f_n^* \oplus \left(\bigoplus_{l \neq n} \operatorname{Rang} f_l^*\right)$, clearly h_n^* is an embedding.

Now we shall show that for each $n, P_{\mathfrak{U}}$ is $\bigoplus_{l<n} \operatorname{Rang} h_n^* \oplus \bigoplus_{l \geq n} \operatorname{Rang} f_l^*$. Why? Because for each n,

$$\operatorname{Rang} f_n^* \oplus \operatorname{Rang} f_{n+1}^* = \operatorname{Rang} h_n^* \oplus \operatorname{Rang} f_{n+1}^*$$

(so 3.2(b) holds as well as 3.2(a)). Next we shall show that $x =: (x^{[i(0)]})f_0^*$ is as required in (c) (this implies the first clause of (c)):

$$\begin{aligned} x = (x^{[i(0)]})f_0^* &= (x^{[i(0)]})h_0^* + x^{[i(0)]} g_{i(0),i(1)} f_1^* \\ &= (x^{[i(0)]})h_0^* + (x^{[i(1)]})f_1^* \\ &= (x^{[i(0)]})h_0 + (x^{[i(1)]})h_1^* + (x^{[i(2)]})f_2^* \\ &= \sum_{l<n} (x^{[i(l)]})h_l^* + (x^{[i(n)]})f_n^*. \end{aligned}$$

The first term is in $\bigoplus_{l<\omega} \operatorname{Rang} h_l^*$ and the second is in $\varphi_{i(n)}(P_{\mathfrak{U}})$.

3.3. Definition. Suppose $\lambda = \operatorname{cf} \lambda > |R| + |S| + \aleph_0$ ($>$ and not $\geq$, just for simplicity), $\mathcal{S} \subseteq \{\delta < \lambda : \operatorname{cf} \delta = \aleph_0\}$ stationary and non-reflecting, $\{\delta < \lambda : \operatorname{cf} \delta = \aleph_0, \delta \notin \mathcal{S}\}$ stationary.

We say $\langle M_\alpha : \alpha \leq \lambda \rangle$ is very nicely constructed for $\mathcal{S}$ and $\mathcal{K}$ (or for $(\mathcal{S}, \mathcal{K})$) if: (A)–(F) of 2.6; only in (C) is M_α a $\operatorname{cl}(\mathcal{K})$-direct summand of M_β and in (F) the direct summands are from $\mathrm{cl}_{\mathrm{is}}\mathcal{K}$, and for each $M \in \mathcal{K}$, for stationarily many $\alpha \in \lambda\backslash\mathcal{S}$, M appears as one of those direct summands; (G) for $\delta \in \mathcal{S}$, $M_{\delta+1}$ is defined either as in (F) or as in (**) of (H) below:

(H) *if* $(*) A \subseteq \lambda\backslash\mathcal{S}$ is unbounded, for $\alpha \in A$ and $n \in \mathcal{U}$ we have $\alpha < \beta_n(\alpha) \in \lambda\backslash\mathcal{S}$, $\mathrm{IDS}^{M_{\beta_n(\alpha)}}_{M_\alpha}(h_{\alpha,n}, N_n)$ (see Definition 2.8) and $\mathcal{U} \subseteq \omega$ is infinite, *then* (**) for some $\delta \in \mathcal{S}$, we have $\langle \alpha_n : n < \omega \rangle$ such that:

(i) $\alpha_n \in A$, $\beta_n(\alpha_n) < \alpha_{n+1}$, $\delta = \bigcup_{n<\omega} \alpha_{n+1}$.

(ii) $M_{\delta+1}$ is defined as in the proof of 2.6, i.e., $M_{\delta+1}$ is $P_\delta \underset{N^*_\delta}{+} M_\delta$, $N^*_\delta = \sum_{n\in\mathcal{U}} h_{\alpha_n,n}(N_n)$, where (using 3.2's notation) P_δ is isomorphic to $P_{\mathcal{U}}$ by an isomorphism h_δ such that the diagram $(n = i(m) = m$th member of $\mathcal{U})$

$$\begin{array}{ccc} N_n & \xrightarrow{h_{\alpha_n,n}} & h_{\alpha_n,n}(N_n) \\ {\scriptstyle h^*_m}\downarrow & & \downarrow{\scriptstyle \mathrm{id}} \\ P_{\mathcal{U}} & \xrightarrow[h_\delta]{} & P_\delta \end{array}$$

commutes and $P_{\delta,n} = (P_{\mathcal{U},n})h$.

So in $M_{\delta+1}$, $P_\delta \cap M_\delta = N^*_\delta$.

Now 3.4, 3.5 below tell us we do not lose in comparison with §2 (and 2.13–2.18 apply), only the error term is smaller; for, e.g., countable R, S it disappears (see 3.6).

3.4. Lemma. (1) *If* $\langle M_\alpha : \alpha \leq \lambda \rangle$ *is very nicely constructed for* $\mathcal{S}$ *and* $\mathcal{K}$ *then for every R-endomorphism F of* M_λ, *for some* $n(*) < \omega$, $\alpha(*) \in \lambda\backslash\mathcal{S}$, *we have* $(\mathrm{Pr}^-)^{n(*)}_{\alpha(*)}[F]$ (*see* 2.8(3)).

(2) *In* (1) *in addition: for some* $z \in N^{\mathrm{tr}}_{n(*)}$, $(\mathrm{Pr1})^{n(*)}_{\alpha(*),z}[F]$ (*see* 2.10).

(3) *In* (1) *in addition: for some* $\bar{L}^* = \langle L^*_n : n \geq n(*) \rangle$, *a decreasing sequence of abelian subgroups of* $\varphi_{n(*)}(M_\lambda)$, $L^*_n \subseteq \varphi_n(M_\lambda)$ (*depending on F, of course*), *we have*:

(i) *for every* $n \geq n(*)$ *and* (*bi-*)*homomorphism* $h : N_n \to M_\lambda$, *we have* $(xh)F - z_n h \in L^*_n + \bigcap_l \varphi_l(M_\lambda)$ *where* $z_n = zg_{n(*),n}$, *and* $L^*_n \subseteq \varphi_n(M_\lambda)$;

(ii) $\bar{L}^*$ *is compact for* $(\bar{\varphi}, n(*))$ *in* M_λ; *i.e.*, if $v_l \in L^*_l$ *for* $l \geq n(*)$ (*but* $l < \omega$) then *for some* $v^* \in L^*_{n(*)}$:

$$\text{for every } n \geq n(*) \qquad v^* - \sum_{l=n(*)}^{n} v_l \in \varphi_{n+1}(M_\lambda).$$

(4) *In* (3) *in addition we can have*: $\bar{L}^*$ *is* $(\mathcal{K}, \bar{\varphi})$*-finitary in* M_λ; *which means for some* $m \geq n(*)$, L^*_m *is* $(\mathcal{K}, \bar{\varphi})$*-finitary in* M_λ, *which means* $L^*_m \subseteq \sum_{i<n} K_i + \bigcap_{l<\omega} \varphi_l(M_\lambda)$, *each* K_i *isomorphic to a member of* $\mathcal{K}$, *and* $\sum_{i<n} K_i$ *a* $\mathcal{K}$*-direct summand of* M_α *for* α *large enough* $\in \lambda\backslash\mathcal{S}$.

(5) If, *for* $N \in \mathcal{K}$, *there is no non-trivial* $\bar{L}$ *(which means* $\bigwedge_m L_m \not\subseteq \bigcap_l \varphi_l(N)$*) compact for* $(\bar{\varphi}, n(*))$ *in* N, then *we can use* $L^* = 0$, *i.e.,* $\bigwedge_n L_n^* = \{0\}$† *[occurs for countable* R, S *and usually]*.

(6) *In* (2) *we can add the parallel of* 2.18, *replacing* $\mathrm{End}^{\bar{\varphi},n}_{<\lambda}(M)$ *by*

$$\mathrm{End}^{\bar{\varphi},n}_{\mathrm{cpt}}(M) = \{h \in \mathrm{End}^{\bar{\varphi},n} : \textit{the range of } h \textit{ is compact for } (\bar{\varphi}, n) \textit{ in } M_\lambda\};$$

similarly $\mathrm{End}^{\bar{\varphi},\omega}_{\mathrm{cpt}}$.

PROOF. (1) Same proof as for 2.7 (using 3.2, of course).

(2) By 2.10's proof (the change in the definition of IDS causes no problem).

(3) Using $n(*), \alpha(*), z$ of (2) we let, for every $n \geq n(*)$ (but $< \omega$),

$$L_n^* =: \{xhF - zg_{n(*),n}h : h \text{ is a bimodule homomorphism from } N_n \text{ into } M_\lambda\}.$$

Let $z_l = zg_{n(*),l} \in N_l$ when $n(*) \leq l < \omega$. By $(\mathrm{Pr1})^{n(*)}_{\alpha(*),z}[F]$ we know that

$$L_n^* \subseteq M_{\alpha(*)} + \bigcap_l \varphi_l(M_\lambda)$$

and easily $L^*_{n(*)}$ is an additive subgroup of $\varphi_{n(*)}(M_\lambda)$.

Clearly (i) holds (by definition of L_n^*), and let us prove (ii). Suppose $v_l^* \in L_l^*$ for $n(*) \leq l < \omega$, so for some $h_l : N_l \to M_\lambda$ a bimodule homomorphism, $v_l^* = (xh_l)F - z_l h_l$ and let $\alpha(0) < \lambda$ be such that $\alpha(0) \notin \mathcal{S}$, $F''(M_{\alpha(0)}) \subseteq M_{\alpha(0)}$, $\mathrm{Rang}\, h_l \subseteq M_{\alpha(0)}$ and $\alpha(0) > \alpha(*)$.

Now note:

(*) for each $n \in (n(*), \omega)$ and $\beta \in \lambda \backslash \mathcal{S}$ for some $\gamma, \beta < \gamma \in \lambda \backslash \mathcal{S}$, some K and some embedding $h_{\beta,n} : N_n \to M_\gamma$ we have:

$$M_\gamma = M_\beta \oplus \mathrm{Rang}\, h_{\beta,n} \oplus K, \qquad K \in \mathrm{cl}(\mathcal{K}), \quad F''(M_\gamma) \subseteq M_\gamma$$

and $x^{[n]} h_{\gamma,n} F \in (\mathrm{Rang}\, h_{\gamma,n}) \oplus K$.

So by choice of $\alpha(*)$, $x^{[n]} h_{\gamma,n} F - z_n h_{\gamma,n} \in \bigcap_{l<\omega} \varphi_l(M_\lambda)$.

[PROOF OF (*). For every $\gamma, \gamma > \beta$, $\gamma \in \lambda \backslash \mathcal{S} \backslash \alpha(0)$, let $h_\gamma : N_n \to M_{\gamma+1}$ and K_γ^0 be such that: h_γ is an embedding and $M_{\gamma+1} = M_\gamma \oplus \mathrm{Rang}\, h_\gamma \oplus K_\gamma^0$; let $\epsilon_\gamma > \gamma$ be in $\lambda \backslash \mathcal{S}$ such that F maps M_{ϵ_γ} into M_{ϵ_γ}; and let, for $\epsilon(1) < \epsilon(2) < \lambda$, $\epsilon(1) \notin \mathcal{S}$,

$$M_{\epsilon(2)} = M_{\epsilon(1)} \oplus K_{\epsilon(1),\epsilon(2)};$$

so $M_{\epsilon_\gamma} = M_\gamma \oplus \mathrm{Rang}\, h_\gamma \oplus K_\gamma^0 \oplus K_{\gamma+1,\epsilon_\gamma}$, and let $x^{[n]} h_\gamma F = v_\gamma + u_\gamma + w_\gamma$ where $v_\gamma \in M_\gamma$, $u_\gamma \in \mathrm{Rang}\, h_\gamma$ and $w_\gamma \in K_\gamma^0 \oplus K_{\gamma+1,\epsilon_\gamma}$. By Fodor's lemma for some v

†We may have to increase $n(*)$.

for a stationary set of $\gamma \in \lambda \setminus \mathcal{S} \setminus \beta \setminus \alpha(0)$, $v_\gamma = v$; choose $\gamma(1), \gamma(2)$ such that: $\epsilon_{\gamma(1)} < \gamma(2)$, and $\gamma(1), \gamma(2)$ are in this set. Let $\gamma = \epsilon_{\gamma(2)}$, $h_{\beta,n} = h_{\gamma(2)} - h_{\gamma(1)}$,

$$K = K_{\beta,\gamma(1)} \oplus K^0_{\gamma(1)} \oplus K_{\gamma(1)+1,\epsilon_{\gamma(1)}} \oplus K_{\epsilon_{\gamma(1)},\gamma(2)}$$

$$\oplus K^0_{\gamma(2)} \oplus \operatorname{Rang} h_{\gamma(1)} \oplus K_{\gamma(2)+1,\epsilon_{\gamma(2)}}.$$

Now the $\gamma, h_{\beta,n}, K$ we have just defined are as required.]

Let $A = \{\beta : \alpha(0) < \beta \notin \mathcal{S}, \beta < \lambda, F''(M_\beta) \subseteq M_\beta\}$. We know that for each $\beta \in A$ for some $\gamma_\beta > \beta$ and embedding $h_{\beta,n} : N_n \to M_{\gamma_\beta}$, $(*)$ above holds. Let $h'_{\beta,l} = h_{\beta,l} + h_l$ for $\beta \in A$, $l \in \mathfrak{U} \stackrel{\text{def}}{=} \{l : n(*) \le l < \omega\}$. By 2.9(1), $\mathrm{IDS}^{M_{\gamma_\beta}}_{M_\beta}(h'_{\beta,l}, N_l, \mathcal{K})$ for $\beta \in A$, $n(*) \le l < \omega$. Now apply 3.3(H) and get $\delta \in \mathcal{S}$ (and $h_\delta : P_{\mathfrak{U}} \to P_\delta$, etc.) as there; let $\gamma < \lambda$ be such that $F''(M_\gamma) \subseteq M_\gamma$, $\gamma > \delta$. Clearly $M_\gamma = M_{\alpha(0)} \oplus P_\delta \oplus K$ for some bimodule $K \in \mathrm{cl}(\mathcal{K})$ and (h^*_l — from 3.2) by chasing the arrows:

$$(**) \qquad x^{[l]} h^*_l h_\delta F = x^{[l]} h'_{\alpha_l,l} F \quad \text{and} \quad z_l h^*_l h_\delta = z_l h'_{\alpha_l,l}$$

and (by the choice of $h'_{\beta,l}$ and by the choice of $h_{\beta,l}$):

$$(***) \qquad x^{[l]} h'_{\alpha_l,l} F - z_l h'_{\alpha_l,l} \in \left(x^{[l]} h_{\alpha_l,l} F - z_l h_{\alpha_l,l}\right) + \left(x^{[l]} h_l F - z_l h_l\right)$$

$$= \left(x^{[l]} h_{\alpha_l,l} F - z_l h_{\alpha_l,l}\right) + v^*_l \in v^*_l + \bigcap_{i<\lambda} \varphi_i(M_\lambda).$$

Remember $x = x^{[n(*)]} f \in P_{\mathfrak{U}}$ (notation of 3.2's proof, so for $i(l)$ there we use $n(*) + l$).

Let $z' = zf \in P_{\mathfrak{U}}$ (remember $z_l = z g_{n(*),l}$ (for $l \in [n(*), \omega)$)) so noting z is $n(*)$-nice and the construction of $P_{\mathfrak{U}}$ for any $m \in [n(*), \omega)$ we have:

$$x - \sum_{l=n(*)}^{m-1} x^{[l]} h^*_l \in \varphi_m(P_{\mathfrak{U}}),$$

$$z' - \sum_{l=n(*)}^{m-1} z_l h^*_l \in \varphi_m(P_{\mathfrak{U}}).$$

Hence

$$x h_\delta - \sum_{l=n(*)}^{m-1} x^{[l]} h^*_l h_\delta \in \varphi_m(P_\delta) \subseteq \varphi_m(M_\lambda),$$

$$z' h_\delta - \sum_{l=n(*)}^{m-1} z_l h^*_l h_\delta \in \varphi_m(P_\delta) \subseteq \varphi_m(M_\lambda).$$

As F is an R-endomorphism

$$x h_\delta F - \sum_{l=n(*)}^{m-1} x^{[l]} h^*_l h_\delta F \in \varphi_m(M_\lambda),$$

so

$$(xh_\delta F - z'h_\delta) - \sum_{l=n(*)}^{m-1} (x^{[l]} h_l^* h_\delta F - z_l h_l^* h_\delta) \in \varphi_m(M_\lambda).$$

Using a projection Z which is the identity on $M_{\alpha(0)}$ and zero on $K \oplus P_\delta$, by (**) we have $(x^{[l]} h_l^* h_\delta F - z_l h_l^* h_\delta)Z = v_l^*$, so

$$(xh_\delta F - z'h_\delta)Z - \sum_{l=n(*)}^{m-1} v_l^* \in \varphi_m(M_\lambda).$$

Hence $(xh_\delta F - z'h_\delta)Z$ is as required.

(4) By (2) above we can have $L^*_{n(*)} \subseteq M_{\alpha(*)}$ for some $\alpha(*) < \lambda$ (without loss of generality $\notin \mathcal{S}$). Now $M_\alpha \in \mathrm{cl}(\mathcal{K})$ and use 3.4A below.

(5) By 3.4B below (and part (4) of 3.4).

(6) Easy, too.

3.4A. Subfact. If $K = \bigoplus_{i \in I} K_i$ (for R-modules), $L_n \subseteq \varphi_n(K)$ (additive subgroup), $\bar{L} = \langle L_n : n(*) \le n < \omega \rangle$ is decreasing and compact for $(\bar{\varphi}, n(*))$ in K, *then* for some finite $J \subseteq I$ and $m < \omega$:

$$L_m \subseteq \bigoplus_{i \in J} K_i + \bigcap_{l < \omega} \varphi_l(K).$$

Proof of 3.4A. If not, choose by induction on $l \ge n(*)$, v_l, J_l, n_l such that: J_l is a finite subset of $I, J_l \subseteq J_{l+1}$,

$$v_l \in L_{n_l} \Big\backslash \left(\bigoplus_{i \in J_l} K_i + \bigcap_l \varphi_l(K) \right) \quad \text{and} \quad v_l \in \bigoplus_{i \in J_{l+1}} K_i;$$

as in the proof of 2.10 it follows that for some n_{l+1},

$$v_l \notin \bigoplus_{i \in J_l} K_i + \varphi_{n_{l+1}}(K).$$

Then find $v^* \in K$ as in 3.4(3)(ii); so for some finite $J \subseteq I$, $v^* \in \bigoplus_{i \in J} K_i$, and an easy contradiction.

3.4B. Subfact. If $\bar{L}$ is compact for $(\bar{\varphi}, n(*))$ in K (R-modules), $h : K \to K'$ is a homomorphism (as R-modules) and

$$[xh \in \varphi_l(K') \backslash \varphi_{l+1}(K') \Rightarrow (\exists y \in K)\,[xy = yh \wedge y \in \varphi_l(K) \backslash \varphi_{l+1}(K)]],$$

then $h''(\bar{L}) = \langle h''(L_n) : n \rangle$ is compact for $(\bar{\varphi}, n(*))$ in K'.

3.4C. REMARK. (1) We can weaken the assumption to: for some $H:\omega\to\omega$ diverging to infinity

$$l \ge n(*) \ \&\ xh \in \varphi_l(K)\setminus\varphi_{l+1}(K) \Rightarrow (\exists y \in K)$$
$$\left[xh = y \ \&\ y \in \varphi_{n(*)}(K) \ \&\ y \notin \varphi_{H(l)}(K)\right].$$

(2) If h is a projection the above condition holds.

PROOF OF 3.4B. Straightforward.

3.4D. SUBFACT. If $L \subseteq K$, $K = \bigoplus_{i=1}^{n} K^i$ and the projection of L to each K^i is $(\mathcal{K},\bar{\varphi})$-finitary, then so is L in K.

3.5. CLAIM. If $\lambda = \mathrm{cf}\,\lambda > |R| + |S|$, $\mathcal{S} \subseteq \{\delta < \lambda : \mathrm{cf}\,\delta + \aleph_0\}$ does not reflect, $\diamondsuit_{\mathcal{S}}$ *then* there is $\langle M_\alpha : \alpha \le \lambda\rangle$ very nicely constructed.

PROOF. Like 2.6.

3.6. CLAIM. If R, S and every $N \in \mathcal{K}$ has cardinality $< 2^{\aleph_0}$, then

(*) for every $\mathcal{K}$-bimodule M and $L_n \subseteq M$ (for $n < \omega$), if $\langle L_n : n_0 \le n < \omega\rangle$ is $(\bar{\varphi},\omega)$-compact in M, then for some m, $L_m \subseteq \bigcap_{l<\omega}\varphi_l(M)$.

3.7. REMARK. If (*) of 3.6 holds, then in 3.4(3) we can choose $L_{n(*)} = 0$; so the "error term" disappears, i.e., for every endomorphism F of M_λ as an R-module, for some m, $F \restriction \varphi_m/\bigcap_{l<\omega}\varphi_l$ is equal to ${}^{m}H^{z}_{M_\lambda}$.

3.8. REMARK. If R, S has cardinality $< 2^{\aleph_0}$, we have interesting such $\mathcal{K}$'s, e.g., $\mathcal{K}$ the family of finitely generated, finitely presented bimodules.

PROOF OF 3.6, 3.7. Easy.

§4. The first Kaplansky test problem

For this section we make:

4.1. HYPOTHESIS. (1) R is a ring, each φ_n a p.e. formula for R-modules (see 2.4) and, for some R-module M^*,

$$\langle \varphi_n(M) : n < \omega\rangle \text{ is strictly decreasing,}$$

(2) λ as in 2.5 for some $\mathcal{S}$.

4.1A. REMARK. We could use the ZFC version of our theorem from [Sh421], only.

4.2. CONCLUSION. Let λ, $\mathcal{S}$ and R, T, S and $\bar{\varphi}$ be as in 2.6, 2.2 and 2.3, respectively. There is a bi-module M,

$$\|M\| = \lambda = \left|\varphi_n(M)\Big/\bigcap_{l<\omega}\varphi_l(M)\right| \qquad \text{(for each } n\text{)}$$

which has few direct decompositions in the following sense:

(i) If $M = \bigoplus_{i\in J} M_i$, then for all but finitely many $i \in J$,

$$\bigvee_n \left[\varphi_n(M_i) = \bigcap_{l<\omega}\varphi_l(M_i)\right].$$

(ii) Assume $|R| + |S| < 2^{\aleph_0}$; *if* $M = K_\alpha \oplus L_\alpha$ for $\alpha < (|R| + |S| + \aleph_0)^+$ *then* for some $\alpha < \beta$ and n

$$\varphi_n(K_\alpha) + \bigcap_l \varphi_l(M) = \varphi_n(K_\beta) + \bigcap_l \varphi_l(M),$$

(iii) $\mathrm{End}^{\bar{\varphi},\omega}(M)/\mathrm{End}^{\bar{\varphi},\omega}_{(|R|+|S|+\aleph_0)^+}(M)$ has cardinality $\leq |R| + |S| + \aleph_0$.

PROOF. (i) By 3.5, there is $\langle M_i : i \leq \lambda\rangle$ which is very nicely constructed. Let $M = M_\lambda$ as an R-module. Assume $M = \bigoplus_{i\in J} M_i$ is a counterexample. By regrouping without loss of generality $J = \omega$, and $\varphi_n(M_n) \neq \bigcap_{l<\omega}\varphi_l(M_n)$. Let F be the R-endomorphism of M defined by: $F \restriction M_i$ is zero for i even, and the identity on M_i for i odd. Apply 3.4; by 3.4(2) for some z $(\mathrm{Pr1})^{n(*)}_{\alpha(*),z}[F]$. By 3.4(3) we get $\bar{L}^* = \langle L_n : n(*) \leq n < \omega\rangle$ a decreasing sequence of abelian subgroups of $\varphi_{n(*)}(M)$, $L^*_n \subseteq \varphi_n(M)$, $\bar{L}^*$ is $(\bar{\varphi}, n(*))$-compact. By 3.4A for some $k < \omega$ and $m < \omega$:

(a) for every $n \geq k$, $L^*_n \subseteq \sum_{i<n} M_i + \bigcap_{l<\omega}\varphi_l(M)$,

(b) if $n \geq n(*)$, $h : N_n \to M$ then $xhF - z_n h \in L^*_n + \bigcap_l \varphi_l(M)$ where $z_n = z g_{n(*),n}$ (on g — see 2.5).

Now choose n large enough and compare what we get for M_n and M_{n+1} to get a contradiction.

(ii) Remember 3.6.

(iii) Should be easy.

4.2A. REMARK. (1) For any T, S as in 2.1, we get the same conclusion (M a bimodule) if we replace $|R|$ by $|R| + |S|$.

(2) If we omit "$|R| + |S| < 2^{\aleph_0}$", we get by the same proof weaker conclusions: with an "error term" which is included in a finitely generated bimodule.

4.3. Conclusion. (1) There are R-modules M, M_1, M_2 of power λ such that $M \oplus M_1 \cong M \oplus M_2$, $M_1 \not\cong M_2$.

(2) Moreover, $M_1 \equiv_{L_{\infty,\lambda}} M_2$ (note $\|M_1\| = \|M_2\| = \lambda$).

4.3A. Remark. (1) Note conclusion (1) is trivial if we omit the "of power λ" — take M_1, M_2, M_3 free R-modules $\|M\| > \|M_2\| > \|M_1\| \geq |R| + \aleph_0$. So the "moreover" in (2) makes it more interesting.

(2) We can ask more of M in 4.3 (and similarly for the other conclusion). It is obtained as in 4.2 for suitable S.

Proof. (1) *A Stage*: Let T be the subring of R which 1 (the unit) generates. Let S be the ring freely generated by $T \cup \{X, W_1, Y, W_2\}$ except

$$XX = X,$$

$$YY = Y,$$

$$XW_1W_2 = X,$$

$$YW_2W_1 = Y,$$

$$XW_1Y = XW_1, \qquad (1 - X)(1 - Y) = 1 - X, \qquad YX = Y,$$

$$YW_2X = YW_2$$

(to understand these equations see the definition of M^a as a bimodule below).

B Stage: Let M^* be an R-module such that $\langle \varphi_n(M^*) : n < \omega \rangle$ is strictly decreasing; let $M^* \overset{h_i}{\cong} M_i^*$ (R-module), $M^a = \bigoplus_{i<\mu} M_i^*$, $\mu = \kappa^{+2}$, $\kappa = (|R| + |S| + \aleph_0)$. We expand M^a to a bimodule by (for $x \in M^*$)

$$(xh_i)X = \begin{cases} xh_i, & i \geq \kappa, \\ 0, & i < \kappa; \end{cases}$$

$$(xh_i)Y = \begin{cases} xh_i, & i \geq \kappa^+, \\ 0, & i < \kappa^+; \end{cases}$$

$$(xh_i)W_1 = \begin{cases} xh_j & \text{if for some } \alpha,\ i = \kappa + \alpha,\ j = \kappa^+ + \alpha, \\ 0 & \text{otherwise}; \end{cases}$$

$$(xh_i)W_2 = \begin{cases} xh_j & \text{if for some } \alpha,\ i = \kappa^+ + \alpha,\ j = \kappa + \alpha, \\ 0 & \text{otherwise}. \end{cases}$$

So assumption 2.3 holds. Let, e.g., $\mathcal{K}$ be from 3.7; hence 3.5 applies and we get a bimodule, $\mathfrak{A} = M_\lambda$. Let ${}_R\mathfrak{A}$ be $\mathfrak{A}$ as an R-module.

C Stage: So every member of S is an endomorphism of ${}_R\mathfrak{A}$. As $XX = X$ we have ${}_R\mathfrak{A} = {}_RM^1 \oplus {}_RM_1$ where ${}_RM^1 = ({}_R\mathfrak{A})X$, ${}_RM_1 = ({}_R\mathfrak{A})(1-X)$. Similarly ${}_R\mathfrak{A} = {}_RM^2 \oplus {}_RM_2$ where ${}_RM^2 = ({}_R\mathfrak{A})Y$, ${}_RM_2 = ({}_R\mathfrak{A})(1-Y)$.

Now W_1, W_2 provide isomorphisms from M^1 onto M^2, so let ${}_RM =: {}_RM^1 \cong {}_RM^2$.

It suffices to show ${}_RM_1 \not\cong {}_RM_2$.

D Stage: Suppose ${}_RM_1 \cong {}_RM_2$; then there are endomorphisms Z_1, Z_2 of ${}_R\mathfrak{A}$, Z_1 mapping ${}_RM_1$ onto ${}_RM_2$, and ${}_RM^1$ onto ${}_RM^2$, and $Z_1Z_2 = Z_2Z_1 = 1$. It is easy to check that:

$$XZ_1 = XZ_1Y, \qquad YZ_2 = YZ_2X,$$

$$(1-X)Z_1 = (1-X)Z_1(1-Y), \qquad (1-Y)Z_2 = (1-Y)Z_2(1-X).$$

So by 3.4 there are $n(*) < \omega$, $z_1, z_2 \in N^{\mathrm{tr}}_{n(*)}$, such that the equations above hold in the endomorphism ring of the abelian group $\varphi_{n(*)}(M)/\bigcap_l \varphi_l(M)$ for any bimodule M when we replace Z_1, Z_2 by ${}^{n(*)}H^{z_1}_M, {}^{n(*)}H^{z_2}_M$ respectively (and interpret $X, Y \in S$ naturally). This holds in particular for the bimodule M^a we have defined in stage B. But by the equations above we get a one-to-one mapping from $\varphi_{n(*)}(\sum_{i<\kappa^+} M_i^*)/\bigcap_l \varphi_l(\sum_{i<\kappa^+} M_i^*)$ onto $\varphi_{n(*)}(\sum_{i<\kappa} M_i^*)/\bigcap_l \varphi_l(\sum_{i<\kappa} M_i^*)$, an easy contradiction (as they have different cardinalities).

(2) We assume the reader knows about $L_{\infty,\lambda}$ and proof of $\equiv_{\infty,\lambda}$ by a hence and forth argument. In the construction we just use $\mathcal{K}$ such that, for each $\alpha < \lambda$, the following bimodule belongs to $\mathcal{K}$: as an R-module it is $M_\alpha \times M_\alpha$, with X, Y, W_1, W_2 interpreted as the identity. (So we construct in 3.5 and extend $\mathcal{K}$ together.)

Note that $X = Y = W_1 = W_2 = 1$ satisfies all the equations; once we note this the checking does not use anything specific on R, T, S.

We may use more specific properties and then use a fixed $\mathcal{K}$; choose it as follows: $\mathcal{K}_0$ is the set of $N_n, N'_n (n < \omega)$; $\mathcal{K}$ is the set of $N \in K_0$ and, for each $N \in K_0$, the bimodule N^* is in $\mathcal{K}$ where N^* is N as an R-module, but multiplication (from the right) by X, Y, W_1, W_2 is zero. So $|\mathcal{K}| < \lambda$ (in fact it is countable). Let $\mathfrak{A} = \bigcup_{\alpha<\lambda} A_\alpha$ be the representation of $\mathfrak{A}$ (i.e., in 3.5, we get $\langle A_\alpha : \alpha < \lambda \rangle$).

4.4. Claim. Suppose S, as a T-module, is free, say $\{s_\beta : \beta < \alpha\}$ is a free basis.

(1) Let $N_{n,0}$ be the R-submodule of N_n which $\{x, y_i : i < k_{m_n-1}\}$ generates. Then N_n, as an R-module, is the direct sum $\sum_{\beta<\alpha} N_{n,\beta}$, $N_{n,0} \overset{h_\beta}{\cong} N_{n,\beta}$ (as R-modules); for

$y \in N_{n,0}$ we have $yh_\beta = ys_\beta$ and $N_{n,0}$ is the R-module generated freely by $\{y_i : i < k_{m_n-1}\}$ except for the equations, and h_0 is the identity.

(2) Hence $\varphi_n(N_n)/\bigcap_l \varphi_l(N_n)$ (as an additive group and even as a T-module) is the direct sum $\sum_{\beta<\alpha} \varphi_n(N_{n,\beta})/\bigcap_l \varphi_l(N_{n,\beta})$.

(3) If $z \in N_n^{\mathrm{tr}}$, then $z = \sum_i z_i h_i$, $z_i \in N_{n,0} \cap N_n^{\mathrm{tr}} \cap \varphi_n(N_{n,0})$, i.e., $z \in \sum_{i<\alpha} \varphi_n(N_{n,i}) \cap N_n^{\mathrm{tr}}$; so $z = \sum_i z_i s_i$ and ${}^nH^z = \sum_i ({}^nH^{z_i})s_i$; z is n-nice iff each z_i is n-nice.

(4) de^n, S (as subrings of dE^n — see 2.15, 2.16) generate dE^n; moreover, they commute. Each member of dE^n has the form $\sum_i x_i s_i$ ($x_i \in de^n$) and $dE^n = de^n \bigotimes_T S$ and de^n is commutative.

(5) Let I_n be a maximal ideal of de^n (to which 1 does not belong); $D_n = de^n/I_n$, $T' = T/I_n \cap T$, $S' = S/I_n \cap T$. So D_n is a field (so commutative).

Any set of equations on S which has a solution in $\mathrm{End}(M)$ for M as in 4.2 has a solution in $D_n \bigotimes_{T'} S'$.

PROOF. Straightforward.

4.5. CONCLUSION. Suppose:

(a) R is a ring satisfying (2) of Theorem 1.A, T the subring 1 generates (so $T \cong \mathbb{Z}/p\mathbb{Z}$, where p is the characteristic of R which is not necessarily prime).

(b) S is a ring, $(S,+)$ is a free T module (so T is a subring of S).

(c) λ is as in 4.2.

Then we can find an R-module M of power λ, and a homomorphism H of S into $\mathrm{End}(M)$ such that:

(d) $\mathrm{Ker}\, H = \{0\}$.

(e) *If* Γ is a set of equations with parameters in $S, H(\Gamma)$ is solvable in $\mathrm{End}(M)$, *then* for some field D [$p > 0 \Rightarrow D$ of characteristic a prime dividing p], [$p = 0 \Rightarrow D$ of characteristic zero, or prime], we have Γ is solvable in $D \otimes S$.

(f) For $s \in S \setminus \{0\}$, $M(H(s))$, the image of M under $H(s)$ has cardinality λ.

PROOF. Left to the reader.

4.6. CONCLUSION. If S is a ring extending $\mathbb{Z}$, (S,+) free, the assumption 2.3 holds and Γ is a set of equations over S not solvable in $D \bigotimes_{\mathbb{Z}_p} (S/pS)$ when D is a field of characteristic dividing that of R, $\mathbb{Z}_p = \mathbb{Z}/p\mathbb{Z}$ if $p > 0$ and $\mathbb{Z}$ if $p = 0$; *then* for M as in 4.2, Γ is not solvable in $\mathrm{End}(M)$ (with S embedded there naturally).

PROOF. Left to the reader.

4.6A. REMARK. In 4.5, 4.6, if $(S,+)$ is $\aleph_0$-free (or $\aleph_0$-free T-modules) the conclusions are similar.

4.7. CLAIM. There are R-modules, M_1, M_2 (as in 4.2), such that:

$$M_1, M_2 \text{ not isomorphic,}$$

$$M_1 \text{ is isomorphic to a direct summand of } M_2,$$

$$M_2 \text{ is isomorphic to a direct summand of } M_1.$$

PROOF. *A Stage*: Let T be the subring of R which 1 generates. Let S be the ring (with 1, associative but not necessarily commutative) extending T generated by $X_1, X_{-1}, W_1, W_{-1}, Z_1, Z_{-1}$ freely except for the equations (to understand them, see below in stage B).

$(*)_1$ $\tau = 0$ if τ is a term,† $M_D^* \tau = 0$ for M_D^* as defined below in stage B for every field D.

We shall prove S is a free T-module.

Let M be as in 4.2 for T,R,S (and $\lambda,\mathcal{S}$). Let $M_1 = MX_1$, $M_{-1} = MX_{-1}$; so M_1, M_{-1} are R-modules as in 4.2, also $M = M_1 \oplus M_{-1}$ (as $X_1^2 = X_1$, $X_{-1}^2 = X_{-1}$, $X_1 + X_{-1} = 1$, $X_1 X_{-1} = X_{-1} X_1 = 0$ in S). We shall show that M_1, M_{-1} are as required in 4.7 (on M_1, M_2).

Also $Z_1^2 = Z_1$, $Z_1 X_1 = Z_1 = X_1 Z_1$ so $M_1 = M_1(1 - Z_1) \oplus M_1 Z_1$; i.e., $M_1 Z_1$ is a direct summand of M_1. On the other hand $M_{-1} \cong M_1 Z_1$ as W_1 maps M_{-1} into $M_1 Z_1$ (since $X_{-1} W_1 = X_{-1} W_1 Z_1$) and W_{-1} maps $M_1 Z_1$ into M_{-1} (since $X_1 Z_1 W_{-1} = W_{-1} X_{-1}$), and the two maps are inverses of each other because $X_{-1} W_1 W_{-1} = X_{-1}$ and $X_1 Z_1 W_{-1} W_1 = Z_1 = X_1 Z_1$.

Similarly $M_{-1} = M_{-1}(1 - Z_{-1}) \oplus M_{-1} Z_{-1}$, so $M_{-1} Z_{-1}$ is a direct summand of M_{-1} and $M_{-1} Z_{-1}$ is isomorphic to M_1. Hence

$$M_1 \cong M_1(1 - Z_1) \oplus M_{-1}, \qquad M_{-1} \cong M_{-1}(1 - Z_{-1}) \oplus M_1.$$

We are left with $M_1 \not\cong M_{-1}$; if they are isomorphic, then as $M = M_1 \oplus M_{-1}$ (for every n large enough) in dE^n there is a solution to the set of equations (in the unknown Y):

$(*)_2$ $X_1 Y X_{-1} = X_1 Y$,
$X_{-1} Y X_1 = X_{-1} Y$,
$YY = 1$.

We shall get a contradiction by 4.5.

†I.e., in the language of rings, in the variables $X_1, X_{-1}, W_1, W_{-1}, Z_1, Z_{-1}$.

B Stage: Let A_1 $[A_{-1}]$ be the set of even [odd] integers, F the following function:

$$F(i) = \begin{cases} i+1, & i \geq 0, \\ i-1, & i < 0. \end{cases}$$

So F maps A_1 into A_{-1} and A_{-1} into A_1, $A_1 \backslash \mathrm{Rang}(F \restriction A_{-1}) = \{0\}$, $A_{-1} \backslash \mathrm{Rang}(F \restriction A_1) = \{-1\}$. Let D be a ring and T be the subring 1 generates. Let i vary on the integers. Let S_0 be the ring generated freely by $\{X_1, X_{-1}, W_1, W_{-1}, Z_1, Z_{-1}\}$.

We define a right $(D \otimes_T S_0)$-module M_D^* as a D-module $M = \sum Dx_i$, with $(\sum a_i x_i)b = \sum_i (a_i b)x_i$ for $a_i, b \in D$. To define multiplication ($x \in M$, $c \in D \otimes_T S_0$) (as D, S_0 commute in $D \otimes_T S_0$) it is enough to define it for $x = x_i$, s one of the generators of S; so let

$$x_i X_1 = \begin{cases} x_i, & i \in A_1, \\ 0, & i \in A_{-1}; \end{cases} \qquad x_i X_{-1} = \begin{cases} 0, & i \in A_1, \\ x_i, & i \in A_{-1}; \end{cases}$$

$$x_i W_1 = x_{F(i)}; \qquad x_i W_{-1} = \begin{cases} x_{F^{-1}(i)}, & i \in \mathrm{Rang}(F), \\ 0, & i \notin \mathrm{Rang}(F); \end{cases}$$

$$x_i Z_1 = \begin{cases} x_i, & i \in A_1 \cap \mathrm{Rang}\, F, \\ 0, & \text{otherwise}; \end{cases} \qquad x_i Z_{-1} = \begin{cases} x_i, & i \in A_{-1} \cap \mathrm{Rang}\, F, \\ 0, & \text{otherwise}. \end{cases}$$

Of course, it is naturally a $(D \otimes_T S)$-module (see definition of S).

C Stage: There is no problem to check that in M_D^* the equations from $(*)_1$ hold, so it is enough to prove that:

(a) in $D \otimes_T S$ there is no solution to $(*)_2$ (i.e., no such Y) (making S have the same characteristic as D),

(b) S is a free T-module.

Clearly S is a T-module, generated by the set of monomials in $\{X_1, X_{-1}, W_1, W_{-1}, Z_1, Z_{-1}\}$.

Our aim now is to show S is a free T-module and find a free basis.

Now for $l \in \{1, -1\}$, $k \in \mathbb{Z}$, $n \geq 0$, $n \geq -k$, we define an endomorphism $\Upsilon^l_{k,n} = {}_D\Upsilon^l_{k,n}$ of M_D^*:

$$x_i \Upsilon^l_{k,n} = \begin{cases} x_{F^k(i)} & \textit{if } F^{-n}(i) \text{ is well defined, } x_i \in A_l \\ 0 & \textit{otherwise} \end{cases}$$

(it is easy to see that it is an endomorphism of M_D^*) and a monomial $Y_{k,n}^l$ (note: for every monomial τ we let τ^0, the zeroth power, be $1 = \mathrm{id}_{M_D^*}$) and remember $n \geq -k$, so $n + k \geq 0$:

$$Y_{k,n}^l = X_l(W_{-1})^n W_1^{n+k}.$$

The reader can check that $Y_{k,n}^l$ as an endomorphism of M_D^* is equal to $\mathfrak{T}_{k,n}^l$.

We next want to prove that $\{Y_{k,n}^l : n, k \in \mathbb{Z}, n \geq 0, n \geq -k, l \in \{1,-1\}\}$ generates S as a T-module; this is done in the next stage.

D Stage: The set $\{Y_{k,n}^l : n, k \in \mathbb{Z}, n \geq -k$ and $l \in \{1,-1\}\}$ generates S as a T-module.

It is enough to show that for every monomial τ, some equation $\tau = \sum a_{n,k}^l Y_{k,n}^l$ holds in S (where $\{(l,n,k) : a_{n,k}^l \neq 0\}$ is finite, $a_{k,n}^l \in T$); i.e., it holds in the ring of endomorphism of M_D^*. We prove this by induction on the length of the monomial.

If the length is zero, τ is 1; now $1 = X_1 + X_{-1}$ (check in M^*) and $X_l = Y_{0,0}^l$. Hence $1 = Y_{0,0}^1 + Y_{0,0}^{-1}$ as required.

If the length is > 0, by the induction hypothesis it is enough to prove:

(*) *if* $\tau \in \{X_1, X_{-1}, W_1, W_{-1}, Z_1, Z_{-1}\}$

then $Y_{k(*),n(*)}^{l(*)}\tau$ is equal to some $\sum_{l,k,n} a_{k,n}^l Y_{k,n}^l$.

(Note: it is enough to check equality on the generators of M^* – the x_i's.)

Let us check:

Case 1. $Y_{k(*),n(*)}^{l(*)} X_l$ is: zero *if* $[l(*) = l \Leftrightarrow k(*)$ odd$]$,
$Y_{k(*),n(*)}^{l(*)}$ *if* $[l(*) = l \Leftrightarrow k(*)$ even$]$.

Case 2. $Y_{k(*),n(*)}^{l(*)} W_l$ is: $Y_{k(*)+1,n(*)}^{l(*)}$ *if* $l = 1$,
$Y_{k(*)-1,n(*)}^{l(*)}$ *if* $l = -1$, $k(*) + n(*) > 0$,
$Y_{k(*)-1,n(*)+1}^{l(*)}$ *if* $l = -1$, $k(*) + n(*) = 0$.

Case 3. $Y_{k(*),n(*)}^{l(*)} Z_l$ is: $Y_{k(*),n(*)}^{l(*)}$ *if* $n(*) + k(*) > 0$ and $[l(*) = l \Leftrightarrow k(*)$ odd$]$,
$Y_{k(*),n(*)+1}^{l(*)}$ *if* $n(*) + k(*) = 0$ and $[l(*) = l \Leftrightarrow k(*)$ odd$]$,
zero *if* $[l(*) = l \Leftrightarrow k(*)$ even$]$.

E Stage: $\{Y_{k,n}^l : (l,k,n) \in \Theta\}$ generate S freely as a T-module where

$$\Theta = \{(l,k,n) : l \in \{1,-1\}, k \in \mathbb{Z}, n \geq 0, k + n \geq 0\}.$$

Suppose $0 = \Sigma\{a^l_{k,n} Y^l_{k,n} : (l,k,n) \in \Theta\}$ as an endomorphism of $(M^*_D,+)$, where we even allow $a^l_{k,n} \in D$. We shall prove that $a^l_{k,n} = 0$ for every $(l,k,n) \in \Theta$.

If $i \in A_1$, $i \geq 0$ then

$$\begin{aligned} 0 &= x_i\left[\sum_{(l,k,n)\in\Theta} a^l_{k,n} Y^l_{k,n}\right] \\ &= \sum_{(l,k,n)\in\Theta} a^l_{k,n}(x_i Y^l_{k,n}) \\ &= \sum\{a^l_{k,n} x_{i+k} : l = 1, (l,k,n) \in \Theta \text{ and } n \leq i\} \\ &= \sum_{j\geq 0} \left(\sum\{a^1_{k,n} : (1,k,n) \in \Theta, i \geq n, i+k = j\}\right) x_j \\ &= \sum_{j\geq 0} \left(\sum\{a^1_{j-i,n} : i \geq n, (1, j-i, n) \in \Theta\}\right) x_j. \end{aligned}$$

Hence for every $i \in A_1$, $i \geq 0$ and $j \geq 0$

$$(*)^a_{i,j} \qquad 0 = \sum\{a^1_{j-i,n} : n \geq 0, n \leq i \text{ and } n + (j-i) \geq 0\}.$$

Similarly, for $i \in A_{-1}$, $i \geq 0$ (equivalently, $i > 0$ as $i \in A_{-1} \Rightarrow i \neq 0$) and $j \geq 0$ we can prove:

$$(*)^b_{i,j} \qquad 0 = \sum\{a^{-1}_{j-i,n} : n \geq 0, n \leq i \text{ and } n + (j-i) \geq 0\}.$$

Similarly, for $i \in A_1$, $i < 0$

$$\begin{aligned} 0 &= x_i\left[\sum_{(l,k,n)\in\Theta} a^l_{k,n} Y^l_{k,n}\right] \\ &= \sum_{(l,k,n)\in\Theta} a^l_{k,n}(x_i Y^l_{k,n}) \\ &= \sum\{a^1_{k,n} x_{i+k} : (1,k,n) \in \Theta \text{ and } -i > n\} \\ &= \sum_{j<0} \left[\sum\{a^1_{j-i,n} : (1, j-i, n) \in \Theta \text{ and } n < -i\}\right] x_j. \end{aligned}$$

Hence for every $i \in A_1$, $i < 0$ and $j < 0$

$$(*)^c_{i,j} \qquad 0 = \sum\{a^1_{j-i,n} : n \geq 0 \text{ and } n + (j-i) \geq 0 \text{ and } n < -i\}.$$

Similarly, for every $i \in A_{-1}$, $i < 0$ and $j < 0$

$$(*)^d_{i,j} \qquad 0 = \sum\{a^{-1}_{j-i,n} : n \geq 0 \text{ and } n + (j-i) \geq 0 \text{ and } i < -n\}.$$

Choose, if possible, (k, m) such that:

(1) $(1, k, m)$ belongs to Θ,

(2) $a^1_{k,m} \neq 0$,

(3) under (1) + (2), m is minimal.

First assume that m is even; in any case $m \geq 0$. Let $i =: m, j =: i + k$ so $i \in A_1$ (being even), $i \geq 0$ and $j = m + k$ is ≥ 0 as $(1, k, m) \in \Theta$. In the equation $(*)^a_{i,j}$ the term $a^1_{k,m}$ appears in the sum, and for every other term $a^1_{k_1,m_1}$ which appears in the sum, we have $m_1 < m$ (and $k_1 = k$). Hence by (3) above it is zero. So it follows that $a^1_{k,m}$ is zero, contradiction.

If m is odd, we get a similar contradiction using $(*)^c_{i,j}$: let $i = -m - 1, j = i + k$, note $m \geq 0$, hence $i < 0$ and i is even, so $i \in A_1$; in the equation $(*)^c_{i,j}$ the term $a^1_{j-i,n} = a^1_{k,n}$ appears in the sum iff $0 \leq n < -i = m + 1$, and $n + (j - i) = n + k \geq 0$ (but if the latter fails, $a^1_{k,m}$ is not defined), so $a^1_{k,m}$ appears, and if another term $a^1_{k_1,m_1}$ appears then $m_1 < m$ (and $k_1 = k$), hence $a^1_{k_1,m_1} = 0$. Necessarily $a^1_{k,m}$ is zero, contradiction.

So $a^1_{k,n} = 0$ whenever it is defined.

Similarly $a^{-1}_{k,n} = 0$ whenever it is defined (use $(*)^b_{i,j} + (*)^d_{i,j}$). Thus we have finished proviing (b) (i.e. (s, ψ) is a free T-module).

F Stage: In particular, for Y from stage C(a), for some $a^l_{k,n}$:

$$Y = \sum \{ a^l_{k,n} Y^l_{k,n} : n \geq 0 \text{ and } k + n \geq 0 \text{ and } l \in \{1, -1\} \}$$

(with only finitely many $a^l_{k,n}$ being non-zero and $a^l_{k,n} \in D$). Let $n(*) < \omega$ be such that

$$a^l_{k,n} \neq 0 \Rightarrow |k|, n < n(*).$$

Let, for $l = 1, -1$,

$$M^{\text{pos}}_l = \left\{ \sum_{i \geq 0} d_i x_i : d_i \in D \text{ and all but finitely many are zero and } d_i \neq 0 \Rightarrow i \in A_l \right\},$$

$$M^{\text{neg}}_l = \left\{ \sum_{i < 0} d_i x_i : d_i \in D \text{ and all but finitely many are zero and } d_i \neq 0 \Rightarrow i \in A_l \right\}.$$

Clearly, as a D-module (really, a left one)

$$M^*_D = M^{\text{pos}}_1 \oplus M^{\text{pos}}_{-1} \oplus M^{\text{neg}}_1 \oplus M^{\text{neg}}_{-1}.$$

Let $Y^r_l = Y \restriction M^r_l$ for $r \in \{\text{pos}, \text{neg}\}$, $l \in \{1, -1\}$. By $(*)_2$ (in stage A) we know $X_1 Y X_{-1} = X_1 Y$, hence Y maps M^{pos}_1 into M^{pos}_{-1} and M^{neg}_1 into M^{neg}_{-1}; i.e., Y^{pos}_1 is into M^{pos}_{-1}, Y^{neg}_1 is into M^{neg}_{-1}.

Similarly by $(*)_2$ we know $X_{-1}YX_1 = X_{-1}Y$, hence Y maps M_{-1}^{pos} into M_1^{pos} and M_{-1}^{neg} into M_1^{neg}. Also, all those mapping Y_1^{pos}, Y_{-1}^{pos}, Y_1^{neg}, Y_{-1}^{neg} are endomorphisms of D-modules. As $Y^2 = 1$ (again by $(*)_2$) we know on $Y_1^{\text{pos}}, Y_{-1}^{\text{pos}}$ that one is the inverse of the other, so both are isomorphisms onto. Similarly for $Y_1^{\text{neg}}, Y_{-1}^{\text{neg}}$.

Let $M_1^{\text{stp}} = \{\sum_{i>0} d_i x_i : d_i \in D$, all but finitely many d_i's are zero and $d_i \neq 0 \Rightarrow i \in A_1\}$. Clearly M_1^{stp} is a sub-D-module of M_1^{pos}. (So what is the difference between M_1^{stp} and M_1^{pos}? Just $x_0 \in M_i^{\text{pos}}$, $x_0 \notin M_1^{\text{stp}}$).

Let $N = \{\sum_{i>n(*)} d_i x_i : d_i \in D$, all but finitely many are zero and $d_i \neq 0 \Rightarrow i \in A_1\}$.

Let $H^{\text{pos}} : M_1^{\text{stp}} \to M_1^{\text{neg}}$ be defined by $x_i H^{\text{pos}} = x_{-i}$ and $H^{\text{neg}} : M_1^{\text{neg}} \to M_1^{\text{stp}}$ be defined by $x_i H = x_{-i}$. Both are isomorphisms onto and endomorphisms of D-modules. By now we know Y_1^{neg} is an isomorphism from M_1^{neg} onto M_{-1}^{neg}, and also $H^{\text{pos}} Y_1^{\text{neg}} H^{\text{neg}}$ is an isomorphism from M_1^{stp} onto M_{-1}^{pos}. Note

$$M_1^{\text{stp}} \xrightarrow{H^{\text{pos}}} M_1^{\text{neg}} \xrightarrow{Y_1^{\text{neg}}} M_{-1}^{\text{neg}} \xrightarrow{H^{\text{neg}}} M_{-1}^{\text{pos}}.$$

However, by the choice of $n(*)$ and N, computing directly we see that

$$Y_1^{\text{pos}} \restriction N = (H^{\text{pos}} Y_1^{\text{neg}} H^{\text{neg}}) \restriction N.$$

Let N^* be the range of $Y_1^{\text{pos}} \restriction N$ and hence also of $(H^{\text{pos}} Y_1^{\text{neg}} H^{\text{neg}}) \restriction N$. So, as Y_1^{pos} is an isomorphism from M_1^{pos} onto M_{-1}^{pos} and $N \subseteq M_1^{\text{pos}}$, we know N^* is a sub-D-module of M_{-1}^{pos} and M_{-1}^{pos}/N^* is isomorphic to M_1^{pos}/N (as D-modules).

But $H^{\text{pos}} Y_1^{\text{neg}} H^{\text{neg}}$ is an isomorphism from M_1^{stp} onto M_{-1}^{pos} and $N \subseteq M_1^{\text{stp}}$, and it maps N onto N^* (see above), so M_1^{stp}/N is isomorphic to M_{-1}^{pos}/N^*. By the previous paragraph we get $M_1^{\text{stp}}/N \cong M_1^{\text{pos}}/N$.

Now M_1^{pos}/N is a free D-module; $\{x_{2i} + N : 0 \leq 2i \leq n(*)\}$ is a free basis and also M_1^{stp}/N is a free D-module: $\{x_{2i} + N : 0 < 2i \leq n(*)\}$ is a free basis; but the number of generators differ by 1.

Appendix: An alternative older proof

On the Proof of 4.7. We can replace the proof from the first equation of stage F as follows:

Let $b_k^l = \sum_n a_{k,n}^l \in D$; so if $i \in \mathbb{Z}$, $|i| > n(*) + 1$ then

$$x_i Y = \sum_{l \in \{1,-1\}, k \in \mathbf{Z}} b_k^l (x_i Y_{k,n}^l).$$

Checking what is $(x_iY)Y$ when $i \in A_{l(*)}$ and $F^{-n(*)}(i)$ is well defined (e.g., $|i| > n(*) + 1$) (i.e., we know $(x_iY)Y = x_i$ as $Y^2 = 1$, on the one hand, and substituting on the other hand) we see that:

(a) for $l \in \{1, -1\}$ there is a unique $k = k_l$ such that:

$$b_k^l \overset{\text{def}}{=} \sum_n a_{k,n}^l \neq 0.$$

If k_1 is even and k_{-1} is odd, choose large enough even $i < \omega$; then

$$((b_{k_1}^1)^{-1} x_{i-k_1})Y = x_i \quad \text{and} \quad ((b_{k_{-1}}^1)^{-1} x_{i-k_{-1}})Y = x_i$$

contradicting "Y is one to one" which follows from $Y^2 = 1$. So "k_1 is even and k_{-1} is odd" is impossible. Similarly "k_{-1} is even and k_1 is odd" is impossible. If k_1, k_{-1} are even we can get a contradiction using the equation $X_1 Y X_{-1} = X_1 Y$ from $(*)_2$. So k_1, k_{-1} are odd.

Now as $Y^2 = 1$:

(b) $k_1 = -k_{-1}$; let $k(*) = k_1$ and

$$\left(\sum_n a_{k(*),n}^1\right)\left(\sum_n a_{\bar{k}(*),n}^{-1}\right) = 1.$$

Hence

(c) for some non-zero $d_i \in D$, $d_i = d(*)$ for any integer i with $|i| > n(*) + 1$, $x_i Y = d_i x_{F^{k(*)}(i)}$ if i is even, $x_i Y = d_i^{-1} X_{F^{-k(*)}(i)}$ if i is odd.

Note

(d) Y maps M^a and M^b into themselves where $M^a = \{\sum_{i \ge 0} d_i x_i : d_i \in D$ and all but finitely many are zero$\}$,
$M^b = \{\sum_{i<0} d_i x_i : d_i \in D$ and all but finitely many are zero$\}$
and $M = M^a \oplus M^b$ (as D-modules).

Now, as $Y^2 = 1$, $M = \mathrm{Rang}(Y) = M^a Y + M^b Y$. Hence:

(e) Y maps M^a onto M^a and M^b onto M^b.

Note

(f) Y is an automorphism of M as a left D-module.

G Stage: Assume $k(*) \neq 1$. Note also that Y maps M^a onto M^a and

$$M_{n(*)}^{a,1} =: \left\{\sum_{\substack{i \ge n(*) \\ i \text{ even}}} d_i x_i : d_i \in D\right\} \quad \text{onto} \quad M_{n(*)+k(*)}^{a,-1} =: \left\{\sum_{\substack{i \ge n(*)+k(*) \\ i \text{ odd}}} d_i x_i : i \in D\right\}$$

(check directly by (c)).

By $(*)_2$ $X_1YX_{-1} = X_1Y$, hence Y maps $M_0^{a,1}$ into $M_0^{a,-1}$; similarly, as by $(*)_2$ $X_{-1}YX_1 = X_{-1}Y$, clearly Y maps $M_0^{a,-1}$ into $M_0^{a,1}$. As $Y^2 = 1$, also Y maps $M_0^{a,1}$ onto $M_0^{a,-1}$, hence Y is an isomorphism from $M_0^{a,1}$ onto $M_0^{a,-1}$ as left D-modules mapping $M_{n(*)}^{a,1}$ onto $M_{n(*)+k(*)}^{a,-1}$, hence $M_0^{a,1}/M_{n(*)}^{a,1} \cong M_0^{a,1}/M_{n(*)+k(*)}^{a,-1}$ but we easily get a contradiction by computing the dimensions.

What if $k(*) = 1$? Then we use M^b and get a similar contradiction if $k(*) \neq -1$.

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